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On McCoy Trivial Extension Rings

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Abstract: The objective of this paper is to explore the potential inheritance of properties from the ring R to its trivial extension ring. Various types of rings are examined, including the McCoy ring, NC-McCoy ring, and α-Armendariz ring.
Keywords: Armendariz ring, McCoy ring, trivial extension ring, α −skew McCoy, Semiprime ring.

1. INTRODUCTION

In the early 1940s, McCoy established a now well-known result, demonstrating that if two polynomials mutually annihilate over a commutative ring, then each polynomial possesses a non-zero annihilator in the base ring [1]. Subsequently, the concept of McCoy rings was independently introduced [2], [3]. In contemporary terms, McCoy's finding asserts that commutative rings exhibit the McCoy property [4]. Several significant equivalent conditions of McCoy rings are proven, particularly in connection with the polynomial ring $R[x]$, and extensive investigations explore the relationships between the McCoy property and various other standard ring-theoretic properties [5], [6].

The ring extension known as the trivial extension of a ring R by an R-module M is represented as $R \ltimes M$, with addition and multiplication defined coordinate-wise as $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. Nagata introduced this construction in 1962 to facilitate the interaction between rings and their modules. The ring $R \ltimes M$ is also referred to as the idealization of M over R [7]. It has been established that R is a right McCoy ring if and only if the trivial extension $T(R, R)$ is also a right McCoy ring. Analogous results hold for weakened (α,δ)-skew Armendariz ring [8], also for von Neumann regular rings and (weak) quasi-regular rings when M = 0 [9], [10], [11]. A question arises regarding whether the trivial extension ring inherits the properties of the original ring, and the connection between the trivial extension of a ring R by an R-module M and the π-regularity of R is explored [12], [11].

A novel concept, known as π-McCoy, is introduced as a generalization encompassing both McCoy rings and IFP rings [13]. The authors concentrate on extending the McCoy ring concept, leading to the introduction of various related concepts as a generalization of the McCoy rings, including α-skew McCoy rings, α-skew π-McCoy rings, McCoy modules, and linearly McCoy rings [14], [15], [16], [4]. Interestingly, there are rings, such as Armendariz rings, where McCoy rings serve as a generalization [3]. Moreover, certain generalizations of McCoy rings are also considered extensions of Armendariz rings for both rings and modules [14], [17]. The relationship between these concepts and more and the trivial extension ring is investigated [18], [19], [20], [21], [11], [22], [23].

Considering the broad relationships that connect McCoy rings to various other concepts, whether at the level of rings and modules or any other concept, focusing on the relationship of trivial extension rings with specific concepts is justified. This is due to the potential for obtaining significant results, and also because these concepts have wide practical applications, both in algebra and in other fields [24], [25], [26], [27], [28], [29].

This paper focuses on the ability of the trivial extension ring to inherit the same property as the original ring.

The order of the paper is given as follows: section 2 presents some previous concepts and facts that are needed in the rest of the work. The results related to McCoy extension rings are given in section 3.

2. PRELIMINARIES

This section defines the terminology and outcomes which is going to be utilized throughout the paper.

Definition 2.1 [2], [3]:A ring R is said to be right McCoy (respectively left McCoy) if for each pair of non-zero polynomials $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$ then there exists a non-zero element $r \in R$ with $f(x)r = 0$ (respectively $rg(x) = 0$). A ring is McCoy if it is both left and right McCoy.

Definition 2.2 [5]: A ring R is Armendariz if given polynomials $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$ then $ab = 0$ for each coefficient a of $f(x)$ and b of $g(x)$.

Theorem 2.3 [2]:A ring R is McCoy if and only if $R[x]$ is McCoy.

Lemma 2.4 [4]:A ring R is semi-commutative if and only if the following three equivalent statements hold:

(1) Any right annihilator over R is an ideal of R.

(2) Any left annihilator over R is an ideal of R.

(3) For any $a, b \in R\ ab = 0$ implies $aRb = 0$.

Trivial extensions attracted attention when people searched for non-reduced rings which are Armendariz. The paper of Rege and Chhawchharia (1997) seems to be the first to consider the Armendariz property of trivial extensions.

Definition 2.5 [18]:For an (R, R) –bimodule M, the trivial extension of R and M, denoted R ∝ M, is the subring $\{\begin{pmatrix} a & m \\ 0 & a \end{pmatrix} : a \in R, m \in M\}$ of the formal upper triangular ring $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$.

3. EXPLORING MCCOY TRIVIAL EXTENSION RINGS ACROSS DIVERSE RING TYPES: KEY FINDINGS

This section presents the primary findings related to the McCoy property of the trivial extension ring examined across various ring types. Note that H is McCoy R-Module in the sense that for $f \in R[x]$ and $g \in H[x], f(x)g(x) = 0$ implies that there existe $r \in R - \{0\}$, such that $f(x).r = 0$ [18].

Theorem 3.1:

Let R be an integral domain and H an R-Module. Let $S = T(R, H)$ be the idealization of R and H. Then $S[x] = T(R, H)[x]$ is McCoy if and only if H is McCoy R-Module. In particular, if R is an integral domain and H is torsion-free, then $S = T(R, H)$ is McCoy.

Proof:Assume that $S[x] = T(R, H)[x]$ is McCoy. Let $0 \neq f(x) = u_n x^n + \cdots u_0$ be an element in $R[x]$, and $g(x) = v_m x^m + \cdots v_0$ be an element in $H[x]$ satisfying $f(x)g(x) = 0$. We will build these two elements in $S[x] = T[R, H]$

Let $A_P = \begin{pmatrix} u_p & 0 \\ 0 & u_p \end{pmatrix}, \quad 0 \leq p \leq n$, and $B_q = \begin{pmatrix} 0 & v_q \\ 0 & 0 \end{pmatrix}, \quad 0 \leq q \leq m.$

Then $0 \neq F(x) = A_n x^n + \cdots A_0 = \begin{pmatrix} f(x) & 0 \\ 0 & f(x) \end{pmatrix}$

$$0 \neq G(x) = B_m x^m + \cdots B_0 = \begin{pmatrix} 0 & g(x) \\ 0 & 0 \end{pmatrix}$$

These elements are satisfying

$F(x)G(x) = 0$ in $S[x]$

$F(x)\mathrm{G}(x) = \begin{pmatrix} f(x) & 0 \\ 0 & f(x) \end{pmatrix} \begin{pmatrix} 0 & f(x) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & f(x)g(x) \\ 0 & 0 \end{pmatrix} = 0$

Now, we have $f(x)g(x) = 0$, so there exists anon zero elements $r, s \in H - \{0\}$ such that $f(x)r = 0$, and $sg(x) = 0$. Therefore H is McCoy.

Conversely, assume that H is McCoy

Let $0 \neq F(x) = \sum_{p=0}^{n} \begin{pmatrix} a_p & 0 \\ 0 & a_p \end{pmatrix} x^p = \begin{pmatrix} f_1(x) & f_2(x) \\ 0 & f_1(x) \end{pmatrix}$, and

$$0 \neq G(x) = \sum_{q=0}^{m} \begin{pmatrix} \acute{a}_q & \acute{b}_q \\ 0 & \acute{a}_q \end{pmatrix} x^q = \begin{pmatrix} g_1(x) & g_2(x) \\ 0 & g_1(x) \end{pmatrix}$$

Be are elements in $R[x]$, satisfies

$F(x)G(x) = 0$, where

$f_1(x) = \sum_{p=0}^{n} a_p x^p$, $\quad f_2(x) = \sum_{p=0}^{n} b_p x^p$, $\quad g_1(x) \sum_{q=0}^{n} \acute{a}_q x^q$, and $\sum_{q=0}^{m} \acute{b}_q x^q$

Are elements in $[x]$, now

$F(x)\mathrm{G}(x) = \begin{pmatrix} f_1(x) & f_2(x) \\ 0 & f_1(x) \end{pmatrix} \begin{pmatrix} g_1(x) & g_2(x) \\ 0 & g_1(x) \end{pmatrix} \begin{pmatrix} f_1(x)g_1(x) & f_1(x)g_2(x) + f_2(x)g_1(x) \\ 0 & f_1(x)g_1(x) \end{pmatrix} = 0$

We have

1- $f_1(x)g_1(x) = 0$

2- $f_1(x)g_2(x) + f_2(x)g_1(x) = 0$

If $f_1(x)g_1(x) = 0$ in $[x]$, since R is a domain. so either $f_1(x) = 0$ or $g_1(x) = 0$.

Say $f_1(x) = 0$, we get$f_1(x)g_2(x) + f_2(x)g_1(x) = f_2(x)g_1(x) = 0$, i.e

$F(x)\, G(x) = \begin{pmatrix} 0 & f_2(x)g_1(x) \\ 0 & 0 \end{pmatrix} = 0$

Since H is McCoy, then there exists $r, s \in R - \{0\}$, such that $f_2(x)\, r = 0$, and $s \,.\, g_1(x) = 0$.

This means there exists an element $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \in S$ satisfying $\begin{pmatrix} f_1(x) & f_2(x) \\ 0 & f_1(x) \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} = 0$, so that $S[x]$ is McCoy. If $g_1(x) = 0$. The proof is similar.

Theorem 3.2:

Let R be a ring. Then R is the right nilpotent coefficient McCoy (NC- McCoy) if and only if the trivial extension $T(R, R)$ is the right nilpotent coefficient McCoy (NC- McCoy) ring.

Proof:Assume that $T(R, R)$ is the right NC- McCoy ring. let $T(R, R) = \acute{R}$.

$$0 \neq F(x) = \begin{pmatrix} r_0 & m_0 \\ 0 & r_0 \end{pmatrix} + \begin{pmatrix} r_1 & m_1 \\ 0 & r_1 \end{pmatrix} x + \cdots \begin{pmatrix} r_n & m_n \\ 0 & r_n \end{pmatrix} x^n$$

$$0 \neq G(x) = \begin{pmatrix} \acute{r}_0 & \acute{m}_0 \\ 0 & \acute{r}_0 \end{pmatrix} + \begin{pmatrix} \acute{r}_1 & \acute{m}_1 \\ 0 & \acute{r}_1 \end{pmatrix} x + \cdots \begin{pmatrix} \acute{r}_m & \acute{m}_m \\ 0 & \acute{r}_m \end{pmatrix} x^m$$

Be two elements in $\acute{R}[x]$ satisfy $F(x)G(x) = 0$.

Define $f_r(x) = r_0 + r_1 x + \cdots r_n x^n$, $f_m(x) = m_0 + m_1 x + \cdots m_m x^m$

$g_r(x) = \acute{r}_0 + \acute{r}_1 x + \cdots \acute{r}_n x^n$, $g_m(x) = \acute{m}_0 + \acute{m}_1 x + \cdots \acute{m}_m x^m$

Are elements in $R[x]$. From $(x)G(x) = 0$, it follows that

$$\begin{pmatrix} f_r(x) & f_m(x) \\ 0 & f_r(x) \end{pmatrix}\begin{pmatrix} g_r(x) & g_m(x) \\ 0 & g_r(x) \end{pmatrix} = \begin{pmatrix} f_r(x)g_r(x) & f_r(x)g_m(x) + f_m(x)g_r(x) \\ 0 & f_r(x)g_r(x) \end{pmatrix} = 0$$

Since $(x)G(x) = 0$. Thus we have

$f_r(x)g_r(x) = 0$ and $f_r(x)g_m(x) + f_m(x)g_r(x) = 0$.

We have nine cases to prove that

1- Let $f_r(x) \neq 0, f_m(x) \neq 0, g_r(x) \neq 0$, and $g_m(x) \neq 0$. Since $f_r(x)g_r(x) = 0$ and also R is right NC-McCoy. Thus there exists a nonzero element $r \in R$ satisfy $f_r(x)\, r \in N(R)[x]$. Hence there exists $0 \neq A = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \in \acute{R}$ satisfy $F(x)A \in N(\acute{R})[x]$ which implies that $\acute{R}$ is right NC-McCoy ring.

2-Let $f_r(x) \neq 0, f_m(x) \neq 0, g_r(x) \neq 0$, and $g_m(x) = 0$. Then we may again choose $0 \neq \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \in \acute{R}$.

3-Let $f_r(x) \neq 0, f_m(x) \neq 0, g_r(x) = 0$, and $g_m(x) \neq 0$. Then we get $f_r(x)g_m(x) = 0$ since R is right NC-McCoy. Then there exists a nonzero element $r \in R$ satisfy $f_r(x)\, r \in N(R)[x]$.Hence there exists $0 \neq A = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \in \acute{R}$ satisfy $F(x)A \in N(\acute{R})[x]$ which implies that $\acute{R}$ is right NC-McCoy ring.

4-Let $f_r(x) \neq 0, f_m(x) = 0, g_r(x) \neq 0$, and $g_m(x) = 0$. Then we get $f_r(x)g_r(x) = 0$ since R is right NC-McCoy. Then there exists a nonzero element $r \in R$ satisfy $f_r(x)\, r \in N(R)[x]$. Hence there exists $0 \neq A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \in \acute{R}$ satisfy $F(x)A = F(x)\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \in N(\acute{R})[x]$. which implies that $\acute{R}$ is right NC-McCoy ring.

5- Let $f_r(x) \neq 0, f_m(x) = 0, g_r(x) \neq 0$, and $g_m(x) = 0$. Then we get $f_r(x)g_r(x) = 0$ since R is right NC-McCoy. Then there exists a nonzero element $r \in R$ satisfy $f_r(x)\, r \in N(R)[x]$.Hence there exists $0 \neq A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \in \acute{R}$ satisfy $F(x)A = F(x)\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \in N(\acute{R})[x]$. which implies that $\acute{R}$ is right NC-McCoy ring.

6- Let $f_r(x) = 0, f_m(x) \neq 0, g_r(x) = 0$, and $g_m(x) \neq 0$. So we may choose the element $0 \neq \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \in \acute{R}$.

The other possibilities are similar to cases (1) – (5).

Conversely, assume that $T(R, R)$ is the right NC-McCoy ring.

7- Let $f(x)\, g(x) = 0$, where

$$f(x) = \sum_{i=0}^{n} r_i\, x^i \quad , \qquad g(x) = \sum_{j=0}^{m} \acute{r}_j\, x^j \quad \in R[x] - \{0\}$$

Where $r_i, \acute{r}_j \in R$. Let $F(x) = \sum_{i=0}^{n} A_i\, x^i$, $G(x) = \sum_{j=0}^{m} B_j\, x^j$

be two elements in $T(R, R)$, where

$A_i = \begin{pmatrix} r_i & 0 \\ 0 & r_i \end{pmatrix}$, $B_j = \begin{pmatrix} r'_j & 0 \\ 0 & r'_j \end{pmatrix} 0 \le i \le n,\ 0 \le j \le m$

So, we have $F(x) \neq 0$ since $f(x) \neq 0$ and $G(x) \neq 0$ since $g(x) \neq 0$

These elements $F(x)$ and $G(x)$ satisfy that $F(x)\ G(x) = 0$ in $T(R,R)$

$$F(x)\ G(x) = \begin{pmatrix} f(x) & f(x) \\ 0 & f(x) \end{pmatrix}\begin{pmatrix} g(x) & g(x) \\ 0 & g(x) \end{pmatrix} = 0$$

Hence, there exists $A = \begin{pmatrix} r & r' \\ 0 & r \end{pmatrix}$ in $T(R,R)$, such that $F(x)\ A = F(x)\begin{pmatrix} r & r' \\ 0 & r \end{pmatrix} \in N\big(T(R,R)\big)[x]$.

Because $A \in T(R,R) - \{0\}$, we have two cases

8- $r \neq 0 8$, which implies that $\begin{pmatrix} r_i r & 0 \\ 0 & r_i r \end{pmatrix}^n \in N(T(R,R)[x]$ Thus, we have $(r_i r)^n = 0\ , 0 \le i \le n$, i.e $(r_i r)^n = (f(x)\ r)^n = 0$, it means $f(x) r \in N(R)[x]$.

9- $r' \neq 0$, which implies that $\begin{pmatrix} 0 & r_i r' \\ 0 & 0 \end{pmatrix}^n \in N(T(R,R)[x]$ Thus, we have $(r_i r')^n = 0\ , 0 \le i \le n$, i.e $(r_i r')^n = (f(x) r')^n = 0$, it means $f(x)\ r' \in N(R)[x]$. Therefore, R is right (NC- McCoy).

Theorem 3.3:

Let α be an endomorphism of a ring R . If R is α-Armendariz ring, then the trivial extension $T(R,R)$ of R is $\overline{\alpha}$ −skew McCoy.

Proof:Assuming R is α-Armendariz, we have to prove that R is α −skew McCoy.

Let $f(x) = a_n x^n + \cdots a_0$, $g(x) = b_m x^m + \cdots b_0 \in R[x,\alpha] - \{0\}$.

Satisfy $f(x)g(x) = 0$, thus $a_i b_j = 0$, for all i and j . Since R is α-Armendariz. Since $f(x) \neq 0$, there exists $a_t \neq 0 \in R$ for some t . Hence

$a_t\ g(x) = a_t b_j = 0$ for all j .Thus R is left α −skew McCoy. Also

From $a_i b_j = 0$, for all i and j

We get $a_i \alpha^i (b_j) = 0$ for all i and j .Since $g(x) \neq 0$, there exists $b_s \neq 0 \in R$ for some s . Hence $a_i \alpha^i (b_s) = 0$ for all i . Thus R is right α −skew, McCoy. Therefore R is $\overline{\alpha}$ −skew McCoy.

Proposition 3.4:

Let R be a semiprime ring. If R is almost Armendariz, then the trivial extension $T(R,R)$ of R is McCoy.

Proof:Assume that R is almost an Armendarize ring. Let $T(R,R) = S$

Let $f_1(x) = \sum_{i=0}^{m} r_i\ x^i$, $f_2(x) = \sum_{i=0}^{m} u_i\ x^i$, $g_1(x) = \sum_{j=0}^{n} s_j\ x^j$, $g_2(x) = \sum_{j=0}^{n} v_j\ x^j$

Be are elements in $R[x]$, we may construct the following elements in $T(R,R) = S$. let

$$F(x) = \begin{pmatrix} r_0 & u_0 \\ 0 & r_0 \end{pmatrix} + \begin{pmatrix} r_1 & u_1 \\ 0 & r_1 \end{pmatrix} x + \cdots \begin{pmatrix} r_m & u_m \\ 0 & r_m \end{pmatrix} x^m \neq 0$$

$$G(x) = \begin{pmatrix} s_0 & v_0 \\ 0 & s_0 \end{pmatrix} + \begin{pmatrix} s_1 & v_1 \\ 0 & s_1 \end{pmatrix} x + \cdots \begin{pmatrix} s_n & v_n \\ 0 & s_n \end{pmatrix} x^n \neq 0$$

Such that $(x)G(x) = 0$, i.e

$$F(x)G(x) = \begin{pmatrix} f_1(x) & f_2(x) \\ 0 & f_1(x) \end{pmatrix}\begin{pmatrix} g_1(x) & g_2(x) \\ 0 & g_1(x) \end{pmatrix} = \begin{pmatrix} f_1(x)g_1(x) & f_1(x)g_2(x) + f_2(x)g_1(x) \\ 0 & f_1(x)g_1(x) \end{pmatrix} = 0$$

It follows that

$f_1(x)g_1(x) = 0 \quad \ldots (1)$ and $f_1(x)g_2(x) + f_2(x)g_1(x) = 0 \ \ldots (2)$

Now there are cases to prove:

(1) If $f_1(x) \neq 0, f_2(x) \neq 0, g_1(x) \neq 0$, and $g_2(x) \neq 0$, then $f_1(x)g_1(x) = 0$. So since R is an almost Armendariz ring, i.e we have

$r_i s_j \in P(R), 0 \le i \le m, 0 \le j \le n$, since R is semiprime ring this implies that $r_i s_j = 0$, for all i and j (from the definition of semiprime). It is clear that there exists $a, b \in R - \{0\}$ such that $f_1(x)\ a = 0$

and $b\ g_1(x) = 0$. Set $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ in S. Then we have $F(x)\ A = 0$ and $B\ G(x) = 0$, therefore, S is McCoy

(2) If $f_1(x) \neq 0, f_2(x) \neq 0, g_1(x) \neq 0$, and $g_2(x) = 0$, then $f_1(x)g_1(x) = 0$. Since R is almost Armendarize (i.e, we have $r_i s_j \in P(R)$, $0 \leq i \leq m$, $0 \leq j \leq n$. because R is a semiprime, then $r_i s_j = 0$ for all i and j. It is clear that there exists $a, b \in R - \{0\}$ such that $f_1(x)\ a = 0$ and $b\ g_1(x) = 0$. Set $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ in S. Then we have $F(x)\ A = 0$ and $B\ G(x) = 0$, therefore, S is McCoy

If $f_1(x) \neq 0, f_2(x) \neq 0, g_1(x) = 0$, and $g_2(x) \neq 0$. then $f_1(x)g_1(x) = 0$, substituting $g_1(x) = 0$ in eq.(2) we get $f_1(x)g_2(x) = 0$ Since R is almost Armendarize (i.e, we have $r_i v_j \in P(R)$, $0 \leq i \leq m$, $0 \leq j \leq n$. because R is a semiprime, thus $r_i v_j = 0$, for all i and j. It is clear that there exists $a, b \in R - \{0\}$ such that $f_1(x)\ a = 0$ and $b\ g_1(x) = 0$. Set $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ in S. Then we have $F(x)\ A = 0$ and $B\ G(x) = 0$, therefore, S is McCoy.

Corollary 3.5:

Let R be a 2-primal ring. If R is weak Armendariz, then the trivial extension $T(R, R)$ is McCoy ring.

Proof:Suppose that R is weak Armendariz, then R is almost Armendariz, and by the above theorem, we get the trivial extension $T(R, R)$ is McCoy ring.

Proposition 3.6:

Every central reduced trivial extension ring $T(R, R)$ of a ring R is McCoy.

Proof:Assume that the trivial extension $T(R, R)$ is a central reduced ring, thus R is a commutative ring and so we have that $T(R, R)$ is McCoy ring.

The converse of Proposition 3.1 is not true in general as shown by the following example:

Example 3.7:

Consider the ring
$S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a \equiv d\ (mod\ \ 2), b \equiv c \equiv 0\ (mod\ \ 2) \right\}$.
In addition, S can be regarded as a trivial extension ring according to the fact that c=0 mod 2.
Let $l_1 = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$ is a nilpotent element of S but this element is not central because, let $l_2 = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \in S$, so $l_1 l_2 = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \neq \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} = l_2 l_1$. Therefore S is not a central reduced ring.
To prove that S is a McCoy ring.
Let $0 \neq f(x) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} x$, $0 \neq g(x) = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$, such that $f(x)g(x) = 0$
there exists an element $r = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \in S - \{0\}$ such that $f(x)r = 0$, thus S is the right McCoy ring, and there exists an element $\acute{r} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \in S - \{0\}$ such that $\acute{r}\, g(x) = 0$, thus S is the left McCoy ring. Therefore S is a McCoy ring.

Remark 3.8:

1. A reduced ring is central reduced.
2. A reduced ring is reversible. It is known that the reversible ring is McCoy. Thus a reduced ring is McCoy but the converse is not true (see Example3.7 and using (1) above).

Proposition 3.9:

Let R be a semi-prime ring. If R is a semi-commutative ring, then the trivial extension $T(R,R)$ of R is a McCoy ring.

Proof:

Suppose that R is a semi-commutative ring. Thus R is reduced,and we get that the trivial extension ring $T(R,R)$ is an Armendariz ring. Therefore, the trivial extension $T(R,R)$ is a McCoy ring.

We call a ring Ra weakly semi-commutative ring if, for any $a, b \in R, ab = 0$ implies arb is a nilpotent element for any $r \in R$. Semi-commutative rings are weakly semi-commutative.

Example 3.10:

Let R be a ring, and let $T = \left\{\begin{pmatrix} c & d \\ 0 & c \end{pmatrix}, a, b \in R\right\}$ be the 2-by-2 upper triangular matrix ring over R, we have the ring T is a weakly semi-commutative.

Let $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in T$, such that $a\ b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$

Let $r = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in T$, then $a\ r\ b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in N(T)$.

We have to prove that T is not a McCoy ring.

let $0 \neq f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x$, $0 \neq g(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x$ be polynomials in T[x]. Then $f(x)g(x) = 0$ but there is not an element $B_0 \in T - \{0\}$ such that $f(x)B_0 = 0$. Thus T is not the right McCoy ring, furthermore, there is no element $A_0 \in T - \{0\}$ such that $A_0\, g(x) = 0$. Thus T is not left McCoy′s ring. Therefore T is not a McCoy ring.

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