*International Journal of Latest Engineering and Management Research (IJLEMR)*
*ISSN: 2455-4847*
*www.ijlemr.com || Volume 07 - Issue 03 || March 2022 || PP. 28-31*

# A Study of the depth-k $K_n$-cylinder $C(k,n)$ in Book-Embedding

I-Fan Sun[1], Shih-Chi Shen[2*]
[1]*Department of Healthcare Administration, I-Shou University, R.O.C.*
[2]*Department of Finance, I-Shou University, R.O.C.*

**Abstract:** The book-embedding problem arises in connection with an approach to fault-tolerant VLSI design. An embedding of a simple undirected graph G in a book consists of an ordering of the vertices of G along the spine of the book, in which edges assigned to the same page do not cross. This study is aimed at the depth-k $K_n$-cylinder $C(k,n)$. We improve some results for n=2,3,4.
**Keywords:** cylinder graph, book-embedding, page width.

## 1. Introduction

A book is a set of half-planes (the pages of the book) that share a common boundary line (the spine of the book). An embedding of a simple undirected graph of G (a pair of vertices are connected by at most one edge) in a book consists of an ordering of the vertices of G along the spine (horizontal line)of the book, together with an assignment of each edge of G to a page of thebook, in which edges assigned to the same page do not cross. This problem have application to several areas of theoretical computer science including VLSI design, algorithms, and complexity theory [1][3][4][5][6].

The minimum number of pages in which a graph G can be embedded is its page number, p(G). And the width of a page is the maximum number ofedges that cross any line perpendicular to the spine of the book. The widthof a book embedding, w(G), is the maximum width of any page of the book. The cumulative page width of a book embedding is the sum of the widths ofall the pages. In this study, we aim at the special graphs the *depth-k $K_n$-cylinder $C(k,n)$*.

## 2. Main Result

The depth-k $K_n$ -cylinder $C(k,n)$ is the graph whose vertex-set is theunion of the k sets $V_{i,n} = \{v_{i,1}, v_{i,2}, \dots, v_{i,n}\}$, $1 \le i \le k$,and whose edges (a) connect each set $V_{i,n}$ into an n-clique, and (b) connecteach vertex $v_{i,j}$ to vertex $v_{i+1,j}$, $1 \le i < k$,$1 \le j < n$. The following resultsis known [2].

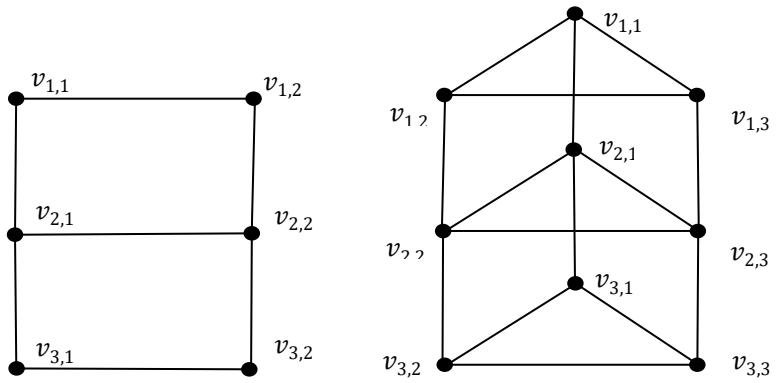



Figure 1: (i) The depth-3$K_2$-cylinder $C(3,2)$, (ii) The depth-3$K_3$-cylinder $C(3,3)$.

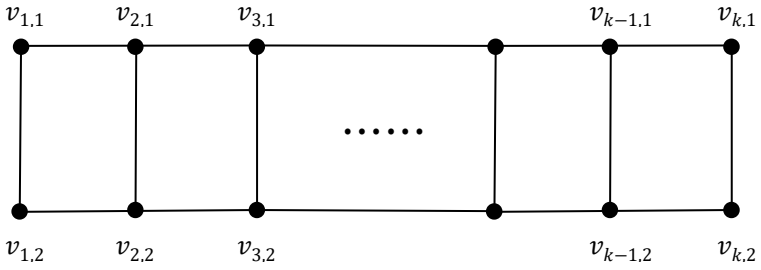
**Proposition 1**. [2]
(1a) Any 1-page layout of $C(k,2)$ has page width at least$\frac{k}{2}$. (1b) There are 2-page layouts of$C(k,2)$ having page width 2.
(2a) Any 2-page layout of $C(k,3)$ has page width at least $\frac{k}{2}$. (2b) Thereare 3-page layouts of $C(k,3)$ having page width 4.
(3) There are 3-page layouts of $C(k,4)$ having page width 4.

In what follows, we improve (1a) and (2) of the Proposition 1.

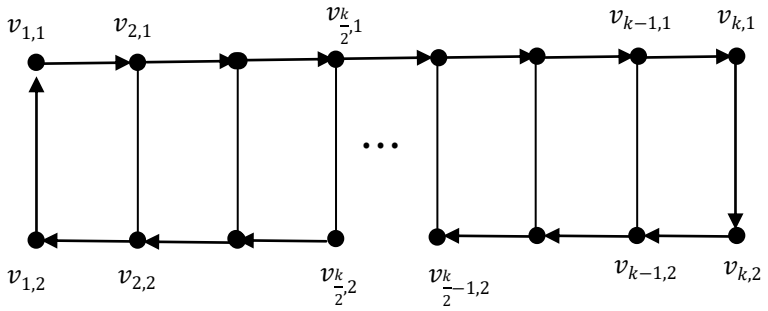
**Proposition 2.** 1-page layouts of $C(k, 2)$ have pagewidth $\left\lceil \frac{k}{2} \right\rceil + 1$.

**Proof.** First, $\left\lceil \frac{k}{2} \right\rceil + 1$ is an upper bound of 1-page layout of $C(k, 2)$ asindicated in Figure 2 (i), (ii) and (iii).

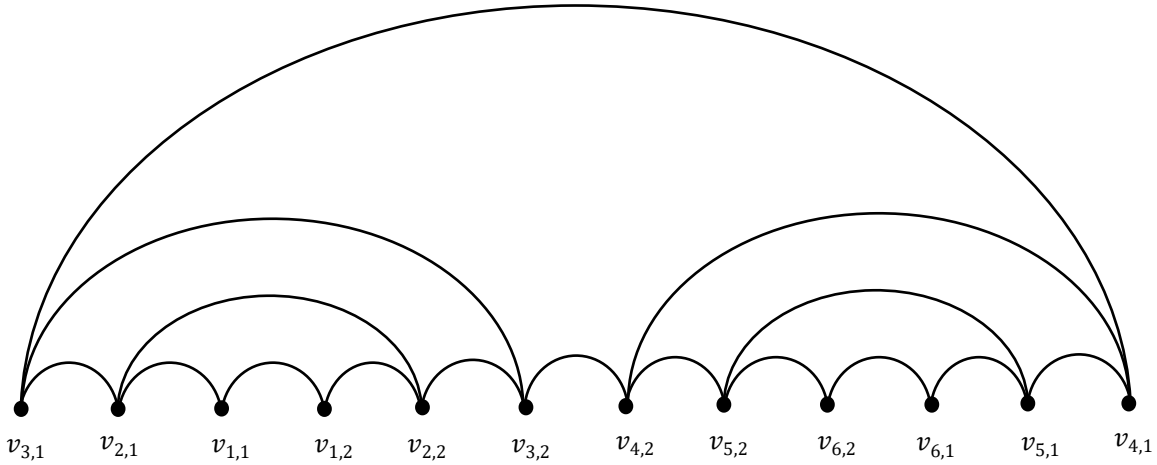
The next step is to prove this embedding is optimal. We check the hamiltonian cycle in $C(k, 2)$ (see Figure 2 (i). The orderingof the vertices which are on the spine must be around the hamiltonian cycle. So, this restricts any ordering of 1-page embedding of $C(k, 2)$. It's not diffcult to see that the ordering indicated in Figure 2 (iii) with the least page width amounts this kind orderings.




(i)




(ii)




(iii)

Figure 2: (i) $C(k, 2)$, (ii) An ordering of the vertices of $C(k, 2)$, (iii) al-page layout of $C(6,2)$ with pagewidth$\left\lceil \frac{k}{2} \right\rceil + 1$.

**Proposition 3.** (a) There are 2-page layouts of $C(k, 3)$ having pagewidth$\left\lceil \frac{k}{2} \right\rceil + 1$. (b) 3-page layouts of $C(k, 3)$ have page width 2.

**Proof**. (a) The result follows immediately from the planar embedding of$C(k, 3)$ is as indicated in Figure 3. And

this embedding gives $\left\lceil \frac{k}{2} \right\rceil + 1$to be an upper bound of the 2-page layout of $C(k, 3)$. (b) The existence ofthe claimed small-page width layouts can be verified by the reader from theillustrative layouts depicted in Figure 4.

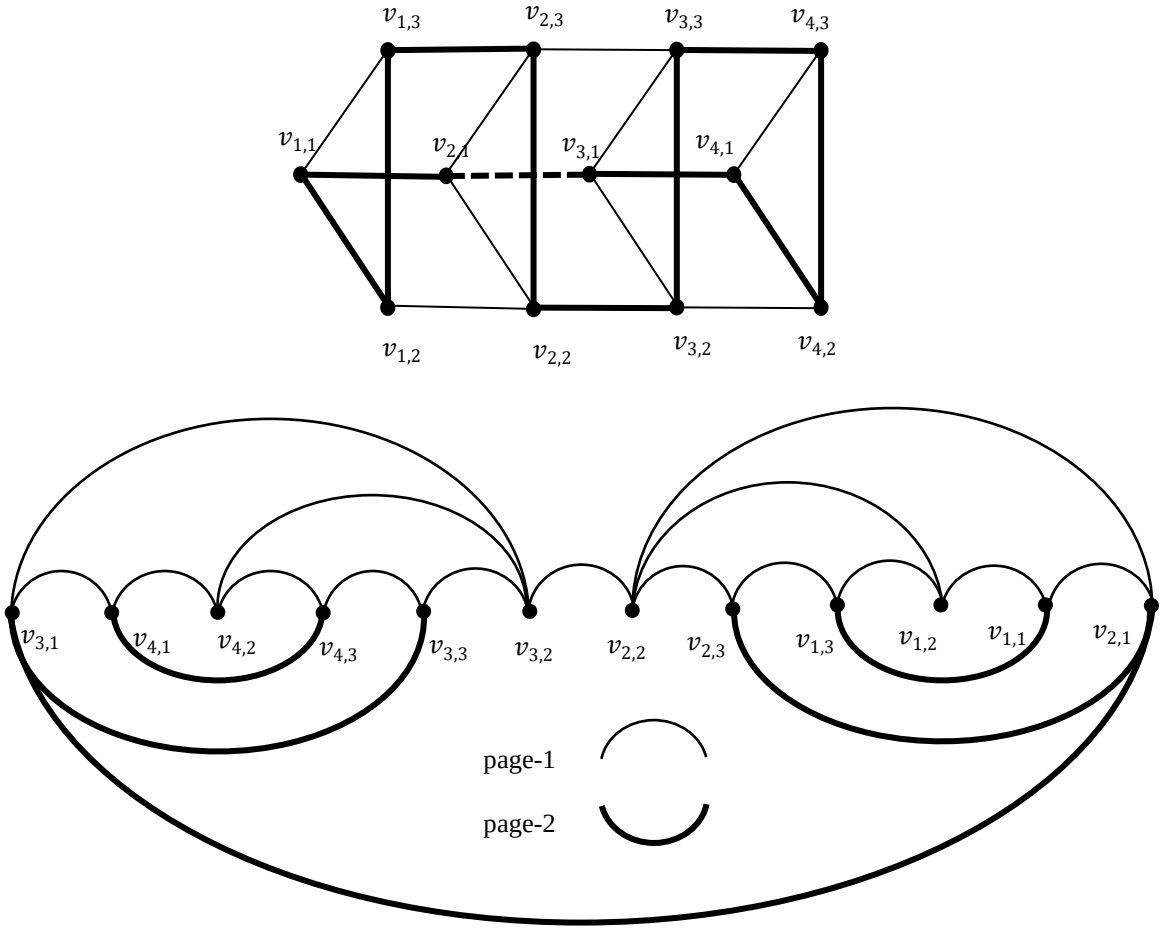



Figure 3 : The unique planar embedding of $C(4,3)$ and its efficient hamiltonian.

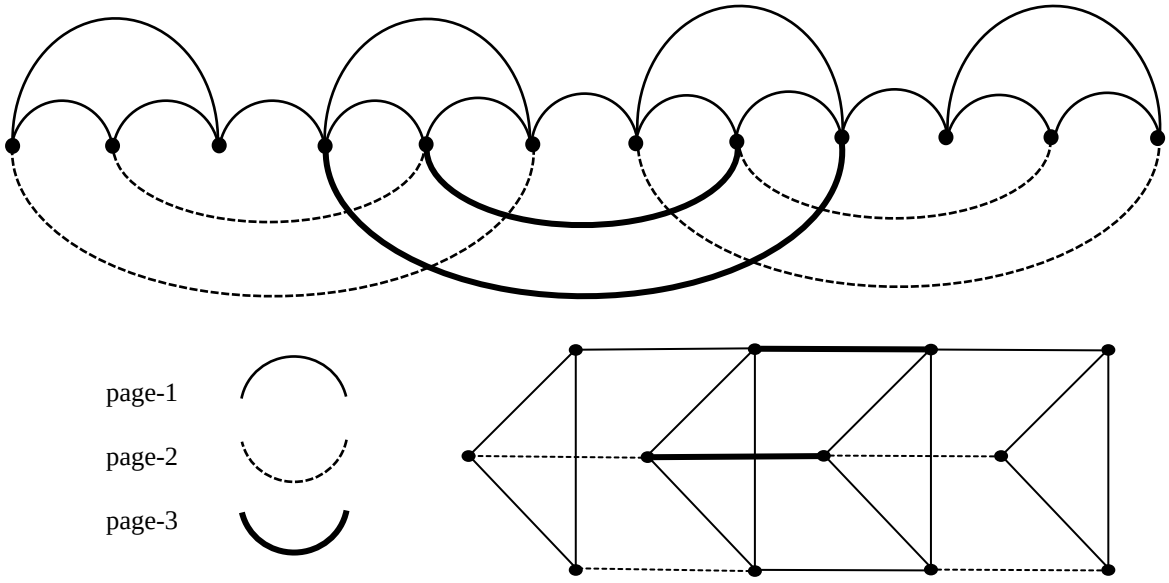



Figure 4 : A 3-page layout of $C(4,3)$ with pagewidth 2.

## References

[1] N. Baudru & S. Fratani, On exteriority notions in book embeddings and treewidth, *Dsicrete Mathematics, 343(4)*, 2020, 111703.

[2] F. R. K. Chung, F. T. Leighton, & A. L. Rosenberg, Embedding graphs in books: A layout problem with applications to VLSI design, *SIAM J Algebraic Discrete Methods 8*, 1987, 33-58.

[3] F. R. K. Chung, F. T. Leighton, & A. L. Rosenberg, DIOGENES: A methodology for designing fault-tolerant processor arrays, *13th Intl Conf. on Fault-Tolerant Computing*, 1983, 26-32.

[4] S.-H. Hong & H. Nagamochi, Simpler algorithms for testing two-page book embedding of partitioned graphs, *Theoretical Computer Science, 725*, 2018, 79-98.

[5] P. Rosenstiehl & R. Tarjan, Gauss codes, planar hamiltonian graphs, and stack-sortable permutations, *J. Algorithms, 5*, 1984, 375-390.

[6] R. Tarjan, Sorting using networks of queues and stacks, *J. Assoc. Comput. Mach.*, *19*, 1972, 341-346.