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Reverse Symmetric Left $*$- n-Multiplier with Involution

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Abstract: In this paper, the symmetric left (right) reverse $*$-n-multiplier and reverse Jordan $*$-n-multiplier are presented and studied. Further, the commutativity and some related results in $*$-ring are considered.
Keywords: prime ring, semiprime ring, multiplier, Jordan centralizer, reverse multiplier

I. INTRODUCTION

Throughout this paper $\mathcal{R}$ will represent an associative ring with center $Z(\mathcal{R})$. For any $u, y \in \mathcal{R}$, the commutator $u\gamma$-γu is denoted by $[u, y]$ and the anti-commutator $u \circ y$ is denoted by uy+ yu [1]. A ring $\mathcal{R}$ is said to be n-torsion free if na=0 with $a \in \mathcal{R}$ then a=0, where n is nonzero integer [2]. Recall that a ring $\mathcal{R}$ is said to be prime if $a\mathcal{R}b$=0 implies that either a=0 or b=0 for all $a, b \in \mathcal{R}$, and it is semiprime if $a\mathcal{R}a$=0 implies that a=0 for all $a \in \mathcal{R}$ [3]. An additive mapping $u \to u^*$ of $\mathcal{R}$ into itself is called an involution if the following conditions are satisfied (i) $(uy)^*$=y^*u^* (ii) $(u^*)^*$= u for all $u, \gamma \in \mathcal{R}$ and $\mathcal{R}$ is called a $*$-ring [4]. In [5], Zalar introduced the term of centralizer (multiplier) and the author proved many results concerning multiplier; an additive mapping $\mathcal{M}$ is called left (resp. right) multiplier if $\mathcal{M}(uy) = \mathcal{M}(u)y$ (resp.$\mathcal{M}(uy\ ) = u\mathcal{M}\ (y\ )$) holds for all $u, y \in \mathcal{R}$, and $\mathcal{M}$ is called a multiplier if it is a left and right multiplier. Further, an additive mapping $\mathcal{M}: \mathcal{R} \to \mathcal{R}$ is called a left (resp. right) Jordan multiplier in case that $\mathcal{M}(u^2)$ = $\mathcal{M}(u)u$ (resp. $\mathcal{M}(u^2)$ = $u\,\mathcal{M}(u)$) holds for $u \in \mathcal{R}$ [6]. In 1991, Zalar proved that for a 2-torsion free semiprime ring every left (right) Jordan multiplier is a left (right) multiplier. An additive mapping $u \to u*$ satisfying ($\mathcal{M}(uy)$*=$\mathcal{M}(u)y*$ (resp.$\mathcal{M}(uy)$*= u*$\mathcal{M}\ (y)$) for all $u, y \in \mathcal{R}$ is called $*$-multipliers. A left (right) Jordan multiplier is an additive mapping $\mathcal{M}: \mathcal{R} \to \mathcal{R}$ which satisfies $\mathcal{M}(u^2)$ = $\mathcal{M}(u)u*$ (resp. $\mathcal{M}(u^2) = u*\ \mathcal{M}(u))u, y \in \mathcal{R}$. In [7], the authors were introduced the concept of reverse *-multipliers (centralizer) of *-ring $\mathcal{R}$ is an additive mapping $\mathcal{M}$:$\mathcal{R} \to \mathcal{R}$ which satisfies $\mathcal{M}(u\gamma)$= $\mathcal{M}(\gamma)$υ* for all $u, \gamma \in \mathcal{R}$. Recently there has been a great deal of work done by many authors on this topic on prime rings and semiprime rings, see ([8], [9]). In [10] The notion of a $*$-multiplier of $\mathcal{R}$ was studied. Many authors have proved the commutativity of prime and semiprime rings admitting multiplier ([11], [12], [13]). This paper is organized as follows. Section 2 is devoted to recalling some mathematical preliminaries and fundamental facts of reverse $*$-n-multiplier and reverse Jordan $*$-n-multiplier. Section 3 presents the commutativity and some related results in $*$-ring.

II. PRELIMINARIES

Some definitions and fundamental facts of reverse $*$-n-multipliers and reverse Jordan $*$-n-multiplier.Throughout this paper consider n is a fixed positive integer.

Proposition 2.1 [2]
Let $\mathcal{R}$ be a ring, then for all $u, \gamma, z \in \mathcal{R}$ we have

1- $[u, \gamma z]$= $\gamma[u, z]$+$[u, \gamma]z$
2- $[u\gamma, z]$= $u[\gamma, z]$+$[u, z]\gamma$
3- $u \circ (\gamma z)$=$(u \circ \gamma)z - \gamma[u, z]$=$\gamma(u \circ z)$+$[u, \gamma]z$
4- $(u\gamma) \circ z$=$u(\gamma \circ z) - [u, z]\gamma$=$(u \circ z)\gamma$+$u[\gamma, z]$

Definition 2.2 [6]
A map $\mathcal{M}: \mathcal{R} \times \mathcal{R} \times ... \times \mathcal{R} \to \mathcal{R}$ is called permuting (or symmetric) if the equation $\mathcal{M}\,(u_1, u_2, \ldots, u_n)$ =$\mathcal{M}\,(u_{\pi(1)}$,$u_{\pi(2)}$,…,$u_{\pi(n)}$) holds, for all$u_i \in \mathcal{R}$ and for every permutation {π(1), π(2),…, π(n)}.

Lemma 2.3 [7]
Let $\mathcal{R}$ be a semiprime $*$-ring and $\mathcal{M}: \mathcal{R} \times \mathcal{R} \times ... \times \mathcal{R} \to \mathcal{R}$ and $a \in \mathcal{R}$ be fixed element If $a\,u - u\,a \in$ Z($\mathcal{R}$) for all u $\in \mathcal{R}$ then we have $a \in$ Z($\mathcal{R}$).
Now, the concepts of reverse $*$-n-multiplier and reverse Jordan $*$-n-multiplier can be presented to get our main results.

Definition 2.4

An n-additive mapping $\mathcal{M}: \mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R} \to \mathcal{R}$ is said to be left reverse $*$-n-multiplier if the following equations hold for all $u_1\, y, u_2, \ldots, u_n \in \mathcal{R}$:

$\mathcal{M}_1(u_1\, y, u_2, \ldots, u_n) = \mathcal{M}_1(y, u_2, \ldots, u_n)\, u_1{}^*$

$\mathcal{M}_2(u_1, u_2 y, \ldots, u_n) = \mathcal{M}_2(u_1, y, \ldots, u_n) u_2{}^*$

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$\mathcal{M}_n(u_1, u_2, \ldots, u_n y) = \mathcal{M}_n(u_1, u_2, \ldots, y) u_n{}^*$

$\mathcal{M}$ is said to be a symmetric left (resp. right) reverse n-multiplier if all the above equations are equivalent to each other. That is, $\mathcal{M}(u_1\, y, u_2, \ldots, u_n) = \mathcal{M}(y, u_2, \ldots, u_n) u_1{}^*$($\mathcal{M}(u_1\, y, u_2, \ldots, u_n) = y^* \mathcal{M}(u_1, u_2, \ldots, u_n)$) for all $y, u_1, u_2, \ldots, u_n \in \mathcal{R}$.

The following example explains the above definitions:

Example 2.5

Consider $\mathcal{R} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{C} \right\}$, where $\mathbb{C}$ is the ring of complex numbers. Clearly, $\mathcal{R}$ is a non-commutative ring under the usual addition and multiplication of matrices. A map $\mathcal{M}: \mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R} \to \mathcal{R}$ is defined by

$$\mathcal{M}\left(\begin{pmatrix} 0 & a_1 & b_1 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_2 & b_2 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & a_n & b_n \\ 0 & 0 & c_n \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & c_1 c_2 \ldots c_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

, for all $\begin{pmatrix} 0 & a_1 & b_1 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_2 & b_2 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & a_n & b_n \\ 0 & 0 & c_n \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{R}$ such that $\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & c & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$.

Then, $\mathcal{M}$ is a symmetric reverse left $*$-n-multipliersand also it is a reverse right $*$-n-multiplier**.**

Now, the concept of symmetric reverse Jordan$*$-n-multiplier is introduced as the following

Definition 2.6

An n-additive symmetric mapping $\mathcal{M}: \mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R} \to \mathcal{R}$ is said to be a symmetric reverse Jordan $*$- n-multiplier in case :

$\mathcal{M}(u_1{}^2, u_2, \ldots, u_n) = \mathcal{M}(u_1, u_2, \ldots, u_n)\, u_1{}^*$ holds for all $u_1, u_2, \ldots, u_n \in \mathcal{R}$

The following example explains the notion of be a symmetric reverse Jordan $*$- n- multiplier

Example 2.7

Consider the ring $\mathcal{R} = \left\{ \begin{pmatrix} u & y \\ 0 & 0 \end{pmatrix} \mid u, y \in \mathbb{R} \right\}$ where $\mathbb{R}$ is the ring of real numbers. Define $\mathcal{M}: \mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R} \to \mathcal{R}$ by

$\mathcal{M}\left(\begin{pmatrix} u_1 & u_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_2 & y_2 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} u_n & y_n \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & u_1 u_2 \ldots u_n \\ 0 & 0 \end{pmatrix}$, for all $\begin{pmatrix} u_1 & u_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_2 & y_2 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} u_n & y_n \\ 0 & 0 \end{pmatrix} \in \mathcal{R}$.

Further, $*$ is defined by $\begin{pmatrix} u & y \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix}$, this means that $\mathcal{M}$ is a symmetric reverse Jordan left$*$ n- multiplier.

III. THE MAIN RESULTS

In [7] and [13], many results of symmetric reverse *-multiplier of prime and semiprime ring with involution are proved. In this paper, these results are studied by using the concept of symmetric reverse *- n-multipliers on $\mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R}$.

We begin by generalizing the following [14, Theorem 2.1] to use some of the results of this paper:

Theorem 3.1

Let $\mathcal{R}$ be a 2-torsion free semiprime ring. If $\mathcal{M}: \mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R} \to \mathcal{R}$ is a n-additive mapping such that $\mathcal{M}(u\gamma u, u_2, \ldots, u_n) = \mathcal{M}(u, u_2, \ldots, u_n)\gamma u$ for all $\gamma, u, u_2, \ldots, u_n \in \mathcal{R}$. Then $\mathcal{M}$ is a symmetric left n-multiplier on $\mathcal{R}$.

Proof:

By assumption,

$\mathcal{M}(u\gamma u, u_2, \dots, u_n) = \mathcal{M}(u, u_2, \dots, u_n)\, \gamma u.$ … (1)

Substituting $u = u + z$ in Equation (1), then

$\mathcal{M}((u+z)\gamma(u+z), u_2, \dots, u_n) = \mathcal{M}(u, u_2, \dots, u_n)\gamma u + \mathcal{M}(u, u_2, \dots, u_n)\gamma z + \mathcal{M}(z, u_2, \dots, u_n)\gamma u + \mathcal{M}(z, u_2, \dots, u_n)\gamma z.$ … (2)

On the other hand, $\mathcal{M}((u+z)\gamma(u+z), u_2, \dots, u_n) = \mathcal{M}((u\gamma u + u\gamma z + z\gamma u + z\gamma z, u_2, \dots, u_n) = \mathcal{M}(u\gamma z + z\gamma u, u_2, \dots, u_n) + \mathcal{M}(u, u_2, \dots, u_n)\gamma u + \mathcal{M}(z, u_2, \dots, u_n).$ …(3)

Combining Equations (2) and (3) we have

$\mathcal{M}(u\gamma z + z\gamma u, u_2, \dots, u_n) = \mathcal{M}(u, u_2, \dots, u_n)\, \gamma z + \mathcal{M}(z, u_2, \dots, u_n)\, \gamma u$ for all $z, \gamma, u, u_2, \dots, u_n \in \mathcal{R}$. … (4)

Let $z = u^2$ in Equation (4) to get

$\mathcal{M}(u\gamma u^2 + u^2\gamma u, u_2, \dots, u_n) = \mathcal{M}(u, u_2, \dots, u_n)\gamma u^2 + \mathcal{M}(u^2, u_2, \dots, u_n)\gamma u.$ ... (5)

Now, replacing γ by $u\gamma + \gamma u$ in Equation (1) and using it to get

$\mathcal{M}(u(u\gamma + \gamma u)u, u_2, \dots, u_n) = \mathcal{M}(u, u_2, \dots, u_n)u\gamma u + \mathcal{M}(u, u_2, \dots, u_n)\gamma u^2$… (6)

Now, combining Equations (6) and (5) will get

$\mathcal{M}(u^2, u_2, \dots, u_n)\gamma u - \mathcal{M}(u, u_2, \dots, u_n)u\gamma u = 0$ … (7)

Let $\mathcal{A}(u) = \mathcal{M}(u^2, u_2, \dots, u_n) - \mathcal{M}(u, u_2, \dots, u_n)u$, then $\mathcal{A}(u)\gamma u = 0$ … (8)

Replacing γ by $uz\mathcal{A}(u)$ in Equation (8) will get

$\mathcal{A}(u)uz\mathcal{A}(u)u = 0$, hence $\mathcal{A}(u)u\mathcal{R}\mathcal{A}(u)u = 0$ … (9)

Since $\mathcal{R}$ is a semiprime then $\mathcal{A}(u)u = 0$ … (10)

Now, let $u = u + \gamma$ in Equation (10) we have

$0 = \mathcal{A}(u)u + \mathcal{A}(\gamma)u + \mathcal{A}(u)\gamma + \mathcal{A}(\gamma)\gamma$ … (11)

Now, we compute $\mathcal{A}(u+\gamma) =$

$\mathcal{M}(u^2, u_2, \dots, u_n) - \mathcal{M}(u, u_2, \dots, u_n)u + \mathcal{M}(y^2, u_2, \dots, u_n) - \mathcal{M}(y, u_2, \dots, u_n)\gamma + \mathcal{M}(u\gamma + \gamma u, u_2, \dots, u_n) - \mathcal{M}(u, u_2, \dots, u_n)\gamma - \mathcal{M}(y, u_2, \dots, u_n)u$.… (12)

Let $\mathcal{B}(u,\gamma) = \mathcal{M}(u\gamma + \gamma u, u_2, \dots, u_n) - \mathcal{M}(u, u_2, \dots, u_n)\gamma - \mathcal{M}(y, u_2, \dots, u_n)u$Then, we have from Equation (12)

$\mathcal{B}(u,\gamma) + \mathcal{A}(u) + \mathcal{A}(\gamma)$, for all $u, \gamma \in \mathcal{R}$. From Equation (11) implies that

$\mathcal{B}(u,\gamma)u + \mathcal{A}(u)u + \mathcal{A}(\gamma)u + \mathcal{B}(u,\gamma)\gamma + \mathcal{A}(u)\gamma + \mathcal{A}(\gamma)\gamma = 0$

By Equation (10),$\mathcal{A}(u)\gamma + \mathcal{B}(u,\gamma)u + \mathcal{A}(\gamma)u + \mathcal{B}(u,\gamma)\gamma = 0.$ … (13)

Now, let $u = -u$ in Equation (13) we get$\mathcal{A}(u)\gamma + \mathcal{B}(u,\gamma)u - \mathcal{A}(\gamma)u - \mathcal{B}(u,\gamma)\gamma = 0.$ … (14)

Adding Equations (13) with (14) and using the fact that $\mathcal{R}$ is a 2-torsion free semiprime ring we find that

$\mathcal{A}(u)\gamma + \mathcal{B}(u,\gamma)u = 0.$ … (15)

Right multiplication of Equation (15) by $\mathcal{A}(u)$to get $\mathcal{A}(u)\gamma\mathcal{A}(u) + \mathcal{B}(u,\gamma)u\mathcal{A}(u) = 0$.… (16)

From Equation (10) we have,$u\mathcal{A}(u)\gamma u\mathcal{A}(u) = 0$. Then

$u\mathcal{A}(u)\,\mathcal{R}u\mathcal{A}(u) = 0.$ … (17)

Also, $u\mathcal{A}(u) = 0.$ …(18)

From Equation (16) and by using Equation (18) will get

$\mathcal{A}(u)\gamma\mathcal{A}(u) = 0$ that $\mathcal{A}(u)\,\mathcal{R}\mathcal{A}(u) = 0$, then $\mathcal{A}(u) = 0$and this means

$\mathcal{M}(u^2, u_2, \dots, u_n) = \mathcal{M}(u, u_2, \dots, u_n)u$. Therefore, $\mathcal{M}$ is a Jordan left n- multiplier and $\mathcal{M}$ is a left n-multiplier on $\mathcal{R}$.

In Theorem 3.1, Substituting $y = u$, then we obtain the following

Corollary 3.2

Let $\mathcal{R}$ be a 2-torsion free semiprime ring If $\mathcal{M} : \mathcal{R} \times \mathcal{R} \times \dots \times \mathcal{R} \to \mathcal{R}$ is a n-additive mapping such that $\mathcal{M}(u^3, u_2, \dots, u_n) = \mathcal{M}(u, u_2, \dots, u_n)u^2$ for all $u, u_2, \dots, u_n \in \mathcal{R}$. Then $\mathcal{M}$ is a symmetric left n-multiplier on$\mathcal{R}$.

Lemma 3.3

Let $\mathcal{R}$ be a semiprime $*$-ring, $a \in \mathcal{R}$ be a fixed element and $\mathcal{M}(u, u_2, \dots, u_n) = a\,u* + u*a$ satisfy $\mathcal{M}(uoy, u_2, \dots, u_n) = \mathcal{M}(u, u_2, \dots, u_n)oy* = u*o\mathcal{M}(y, u_2, \dots, u_n)$ for all $y, u, u_2, \dots, u_n \in \mathcal{R}$. Then $a \in Z(\mathcal{R})$.

Proof:

$\mathcal{M}(uoy, u_2, \dots, u_n) = \mathcal{M}(u, u_2, \dots, u_n)oy* = u*o\mathcal{M}(y, u_2, \dots, u_n).$ …(1)

By hypothesis $\mathcal{M}(u, u_2, \dots, u_n) = a\,u* + u*a$, one has $\mathcal{M}(uoy, u_2, \dots, u_n) = (a\,u* + u*a)y* + y*(a\,u* + u*a).$ …(2)

Let $u = uy + yu$ in hypothesis relation to get

$\mathcal{M}(uoy, u_2, \dots, u_n) = a(uy + yu)* + (uy + yu)*a.$ … (3)

Then, by Equations (1) and (2), one has

$ay*u*+u*y*a-u*ay*-y*au*=0$
Hence, $(ay*-y*a)u*+u*(y*a-ay*)=0$ and this implies that $[[a,y*],u*]=0$
ApplyingLemma (2.3) will get $a\in Z(\mathcal{R})$.

Lemma 3.4
Let $\mathcal{R}$ be a semiprime$*$-ring. Then, every mappings $\mathcal{M}$ of $\mathcal{R}$ satisfy $\mathcal{M}(uoy,u_{2},\ldots,u_n)=\mathcal{M}(u,u_{2},\ldots,u_n)oy*=u*o\mathcal{M}(y,u_{2},\ldots,u_n)$ for all $y,u,u_{2},\ldots,u_n\in\mathcal{R}$. Then, $\mathcal{M}$ maps $Z(\mathcal{R})$ into $Z(R)$.

Proof:
Suppose that $a=\mathcal{M}(c,u_{2},\ldots,u_n)$ for $c\in Z(\mathcal{R})$ and $u,u_{2},\ldots,u_n\in\mathcal{R}$
$2\mathcal{M}(cu,u_{2},\ldots,u_n)=\mathcal{M}(cu+uc,u_{2},\ldots,u_n)=\mathcal{M}(c,u_{2},\ldots,u_n)u*+u*\mathcal{M}(c,u_{2},\ldots,u_n)=au*+u*a$.
Now, $\mathcal{M}(uy+yu,u_{2},\ldots,u_n)=2\mathcal{M}(c(uy+yu),u_{2},\ldots,u_n)=2\mathcal{M}(cu,u_{2},\ldots,u_n)y*+y*\mathcal{M}(cu,u_{2},\ldots,u_n)$
$=2\mathcal{M}(cy,u_{2},\ldots,u_n)u*+u*\mathcal{M}(cy,u_{2},\ldots,u_n)=\mathcal{M}(cu,u_{2},\ldots,u_n)y*+y*\mathcal{M}(cu,u_{2},\ldots,u_n)$
$=\mathcal{M}(cy,u_{2},\ldots,u_n)u*+u*\mathcal{M}(cy,u_{2},\ldots,u_n)$
$=\mathcal{M}(u,u_{2},\ldots,u_n)y*+y*\mathcal{M}(u,u_{2},\ldots,u_n)$
$=\mathcal{M}(y,u_{2},\ldots,u_n)u*+u*\mathcal{M}(y,u_{2},\ldots,u_n)$, forall $y,u,u_{2},\ldots,u_n\in\mathcal{R}$ By Lemma (3.3), one gets $a\in Z(\mathcal{R})$.

Theorem 3.5
Let $\mathcal{R}$ is a 2-torsion free semiprime $*$-ring, and $\mathcal{M}:\mathcal{R}\times\mathcal{R}\times\ldots\times\mathcal{R}\to\mathcal{R}$ is an n-additive mapping which satisfies $\mathcal{M}(uoy,u_{2},\ldots,u_n)=\mathcal{M}(u,u_{2},\ldots,u_n)oy*=u*o\mathcal{M}(y,u_{2},\ldots,u_n)$ for all $y,u,u_{2},\ldots,u_n\in\mathcal{R}$. Then, $\mathcal{M}$ is a reverse $*$-n-multiplier of $\mathcal{R}$.

Proof:
Notice that$\mathcal{M}(uoy,u_{2},\ldots,u_n)=\mathcal{M}(u,u_{2},\ldots,u_n)oy*\ u*o\mathcal{M}(y,u_{2},\ldots,u_n)\mathcal{M}(uy+yu,u_{2},\ldots,u_n)$
$=\mathcal{M}(u,u_{2},\ldots,u_n)y*+y*\mathcal{M}(u,u_{2},\ldots,u_n)=\mathcal{M}(y,u_{2},\ldots,u_n)u*+u*\mathcal{M}(y,u_{2},\ldots,u_n)$.
Replacing $y=uoy$ in the last relation, one will have
$\mathcal{M}(u,u_{2},\ldots,u_n)(uy+yu)*+(uy+yu)*\mathcal{M}(u,u_{2},\ldots,u_n)$
$=\mathcal{M}(u,u_{2},\ldots,u_n)y*u*+y*\mathcal{M}(u,u_{2},\ldots,u_n)u*+u*\mathcal{M}(u,u_{2},\ldots,u_n)y*+u*y*\mathcal{M}(u,u_{2},\ldots,u_n)$
This implies that $\mathcal{M}(u,u_{2},\ldots,u_n)u*y*+y*u*\mathcal{M}(u,u_{2},\ldots,u_n)=y*\mathcal{M}(u,u_{2},\ldots,u_n)u*+u*\mathcal{M}(u,u_{2},\ldots,u_n)y*$.
Then, $\mathcal{M}(u,u_{2},\ldots,u_n)\,o\,(u\,o\,y)*=(\mathcal{M}(u,u_{2},\ldots,u_n)\,o\,y*)u*$.
Also, one will get , $[\mathcal{M}(u,u_{2},\ldots,u_n),u*]y*=y*[\mathcal{M}(u,u_{2},\ldots,u_n),u*]$
The following is obtained$[\mathcal{M}(u,u_{2},\ldots,u_n),u*]\in Z(\mathcal{R})$.
Now, one will show that$[\mathcal{M}(u,u_{2},\ldots,u_n),u*]=0$,and let $c\in Z(\mathcal{R})$ one gets $2\mathcal{M}(cu,u_{2},\ldots,u_n)=\mathcal{M}(cu+uc,u_{2},\ldots,u_n)=\mathcal{M}(c,u_{2},\ldots,u_n)u*+u*\mathcal{M}(c,u_{2},\ldots,u_n)=2\mathcal{M}(u,u_{2},\ldots,u_n)c*$. By using Lemma (3.4), the result is $\mathcal{M}(cu,u_{2},\ldots,u_n)=\mathcal{M}(u,u_{2},\ldots,u_n)c*=\mathcal{M}(c,u_{2},\ldots,u_n)u*$. Also, for all $c\in Z(\mathcal{R})$, one takes that
$[\mathcal{M}(u,u_{2},\ldots,u_n),u*]c*=\mathcal{M}(u,u_{2},\ldots,u_n)u*c*-u*\mathcal{M}(u,u_{2},\ldots,u_n)c*=\mathcal{M}(u,u_{2},\ldots,u_n)c*u*-u*\mathcal{M}(u,u_{2},\ldots,u_n)c*=\mathcal{M}(c,u_{2},\ldots,u_n)u^{*2}-u*\mathcal{M}(c,u_{2},\ldots,u_n)u*$
$=\mathcal{M}(c,u_{2},\ldots,u_n)u*u*-u*\mathcal{M}(c,u_{2},\ldots,u_n)u*=[\mathcal{M}(c,u_{2},\ldots,u_n),u*]u*$ for all $c\in Z(\mathcal{R})$, also one gets $\mathcal{M}(c,u_{2},\ldots,u_n)\in Z(\mathcal{R})$, then $=\mathcal{M}(c,u_{2},\ldots,u_n)u*u*-\mathcal{M}(c,u_{2},\ldots,u_n)u*u*=\mathcal{M}(c,u_{2},\ldots,u_n)u^{*2}-\mathcal{M}(c,u_{2},\ldots,u_n)u^{*2}$
On other hand, one will show that,
$2\ \mathcal{M}(u^2,u_{2},\ldots,u_n)=\mathcal{M}(uu+uu,u_{2},\ldots,u_n)=\mathcal{M}(u,u_{2},\ldots,u_n)u*+u*\mathcal{M}(u,u_{2},\ldots,u_n)=2\ u*\mathcal{M}(u,u_{2},\ldots,u_n)=2\mathcal{M}(u,u_{2},\ldots,u_n)u*$

Theorem 3.6
Assume that $\mathcal{R}$ be a 2-torsion free semiprime ring with an identity element, $\mathcal{M}:\mathcal{R}\times\mathcal{R}\times\ldots\times\mathcal{R}\to\mathcal{R}$ be an n-additive mapping such that $\mathcal{M}(u^3,u_{2},\ldots,u_n)=u*\mathcal{M}(u,u_{2},\ldots,u_n)u*$. Then, $\mathcal{M}$ is a reverse $*$-n-multiplier, that is $\mathcal{M}(uy,u_{2},\ldots,u_n)=\mathcal{M}(y,u_{2},\ldots,u_n)u*=y*\mathcal{M}(u,u_{2},\ldots,u_n)$ for all $y,u,u_{2},\ldots,u_n\in\mathcal{R}$.

Proof:
Since $\mathcal{M}(u^3,u_{2},\ldots,u_n)=u*\mathcal{M}(u,u_{2},\ldots,u_n)u*$. … (1)

Multiply involution both sides to Equation (1) to get the following $\mathcal{M}(u^3, u_{2,}\dots, u_n) * = u\,\mathcal{M}(u, u_{2,}\dots, u_n) * u$ for all $u\ , u_{2,}, \dots, u_n \in \mathcal{R}$.

Suppose that $F: \mathcal{R} \times \mathcal{R} \times \dots \times \mathcal{R} \to \mathcal{R}$, then one has $F(u, u_{2,}\dots, u_n) = (\mathcal{M}(u, u_{2,}\dots, u_n)) *$, and also we get $F(u^3, u_{2,}\dots, u_n) = (\mathcal{M}(u^3, u_{2,}\dots, u_n)) * = (u * \mathcal{M}(u, u_{2,}\dots, u_n)\, u *) *= u\,(\mathcal{M}(u, u_{2,}\dots, u_n)) * u = u\,F(u, u_{2,}\dots, u_n)\,u$

Now, by using Corollary 3.1 we have F is n-multiplier $F(uy, u_{2,}\dots, u_n) = uF(u, u_{2,}\dots, u_n) = F(u, u_{2,}\dots, u_n)u$. Then, $(\mathcal{M}(uy, u_{2,}\dots, u_n)) *= F(uy, u_{2,}\dots, u_n) = uF(y, u_{2,}\dots, u_n) = u\,(\mathcal{M}(y, u_{2,}\dots, u_n)) *$ for all $y,\ u\ , u_{2,}, \dots, u_n \in \mathcal{R}$. ... (2)

Also,
$(\mathcal{M}(uy, u_{2,}\dots, u_n)) * = F(uy, u_{2,}\dots, u_n) = F(u, u_{2,}\dots, u_n)y = (\mathcal{M}(u, u_{2,}\dots, u_n)) * y$ for all $y,\ u\ , u_{2,}, \dots, u_n \in \mathcal{R}$. ... (3)

Multiply involution both sides to Equations (2) and (3) to get $\mathcal{M}(uy, u_{2,}\dots, u_n) = \mathcal{M}(y, u_{2,}\dots, u_n)u * = y * \mathcal{M}(u, u_{2,}\dots, u_n)$.

Theorem 3.7

Let $\mathcal{R}$ be a semiprime ring, and $\mathcal{M}: \mathcal{R} \times \mathcal{R} \times \dots \times \mathcal{R} \to \mathcal{R}$ be additive mapping such that $\mathcal{M}(u, u_{2,}\dots, u_n)y *= u * \mathcal{M}(y, u_{2,}\dots, u_n)$ for all $y,\ u\ , u_{2,}, \dots, u_n \in \mathcal{R}$.Then, $\mathcal{M}$ is a reverse left $*$-n-multiplier of $\mathcal{R}$.

Proof:

Notice that $\mathcal{M}(u, u_{2,}\dots, u_n)y{*}=u{*}\mathcal{M}(y, u_{2,}\dots, u_n)$. … (1)

Calculating the following equation and by Equation (1), we have

$0=\mathcal{M}(u+y, u_{2,}\dots, u_n)z * - \mathcal{M}(u, u_{2,}\dots, u_n)z * - \mathcal{M}(y, u_{2,}\dots, u_n)z *$

$= (u+y) * \mathcal{M}(z, u_{2,}\dots, u_n) - u * \mathcal{M}(z, u_{2,}\dots, u_n) - y * \mathcal{M}(z, u_{2,}\dots, u_n)$

$= (u+y) * -u * -y * (\mathcal{M}(z, u_{2,}\dots, u_n))$

$= u * +y * -u * -y * (\mathcal{M}(z, u_{2,}\dots, u_n))$

This implies that $(\mathcal{M}(u+y, u_{2,}\dots, u_n) - \mathcal{M}(u, u_{2,}\dots, u_n) - \mathcal{M}(y, u_{2,}\dots, u_n))z *= 0$. … (2)

Now, let $z *= z$ in Equation (2) to get

$(\mathcal{M}(u+y, u_{2,}\dots, u_n) - \mathcal{M}(u, u_{2,}\dots, u_n) - \mathcal{M}(y, u_{2,}\dots, u_n))z=0$ for all $y,\ u\ , u_{2,}, \dots, u_n \in \mathcal{R}$. … (3)

Since $\mathcal{R}$ is semiprime ring, one obtains that $\mathcal{M}(u+y, u_{2,}\dots, u_n) = \mathcal{M}(u, u_{2,}\dots, u_n) + \mathcal{M}(y, u_{2,}\dots, u_n)$. Similarly, one calculates the relation $(\mathcal{M}(uy, u_{2,}\dots, u_n) - \mathcal{M}(u, u_{2,}\dots, u_n)y *)z*$, then$\mathcal{M}$ is a reverse left $*$-n-multiplier of$\mathcal{R}$.

Theorem 3.8

Let $\mathcal{R}$ be a 2-torsion free semiprime ring, and $\mathcal{M}: \mathcal{R} \times \mathcal{R} \times \dots \times \mathcal{R} \to \mathcal{R}$ be a Jordan left $*$-n-multiplier. Then, $\mathcal{M}$ is a reverse left $*$-n-multiplier, which is $\mathcal{M}(uy, u_{2,}\dots, u_n) = \mathcal{M}(y, u_{2,}\dots, u_n)u*$ for all $y, u\ , u_{2,}, \dots, u_n \in \mathcal{R}$.

Proof:

Since $\mathcal{M}(uy, u_{2,}\dots, u_n) = \mathcal{M}(y, u_{2,}\dots, u_n)u *$. ... (1)

Substituting $y = u$ in to Equation (1) and by applying involution the both sides to get the following:

$(\mathcal{M}(u^2, u_{2,}\dots, u_n)) *= u\mathcal{M}(u, u_{2,}\dots, u_n) *$ for all$u\ , u_{2,}, \dots, u_n \in \mathcal{R}$

Suppose that $F: \mathcal{R} \times \mathcal{R} \times \dots \times \mathcal{R} \to \mathcal{R}$, $F(u, u_{2,}\dots, u_n) = (\mathcal{M}(u, u_{2,}\dots, u_n))^*$.

This implies that $F(u^2, u_{2,}\dots, u_n) = (\mathcal{M}(u^2, u_{2,}\dots, u_n)) *$

$= (\mathcal{M}(u, u_{2,}\dots, u_n)\, u *) *= u\,(\mathcal{M}(u, u_{2,}\dots, u_n)) * = u\,F(u, u_{2,}\dots, u_n)$. Thus$F$ is a Jordan right n-multiplier on $\mathcal{R}$.

That is, $F(uy, u_{2,}\dots, u_n) = u\mathcal{M}(y, u_{2,}\dots, u_n)$ for all $y, u\ , u_{2,}, \dots, u_n \in \mathcal{R}$. So, we have, $(\mathcal{M}(uy, u_{2,}\dots, u_n)) * = F(uy, u_{2,}\dots, u_n) = u\,F(y, u_{2,}\dots, u_n) = u(\mathcal{M}(y, u_{2,}\dots, u_n)) *$. ... (2)

Also $(\mathcal{M}(uy, u_{2,}\dots, u_n)) * = u(\mathcal{M}(y, u_{2,}\dots, u_n)) *$

When applying involution to both sides of the above relation, then$\mathcal{M}(uy, u_{2,}\dots, u_n) = \mathcal{M}(y, u_{2,}\dots, u_n)u *$ for all $y, u\ , u_{2,}, \dots, u_n \in \mathcal{R}$.

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