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Lyapunov-Type Inequalities for the Quasilinear $q.\omega$-Difference Systems

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Abstract: Using the Hö lder inequality, we establish several Lyapunov-type inequalities for quasilinear $q.\omega$-difference equations and $q.\omega$-difference systems.

Keywords: Lyapunov-type inequality, $q.\omega$ -difference equations, $q.\omega$ -difference systems, Hö lder inequality.

I. Introduction

The Lyapunov inequality and many of its generalizations have proved to be useful tools in oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications for the theories of differential and difference equation. The well-known inequality of Lyapunov [1] states that a necessary condition for the boundary value problem $y''+q(t)y=0, y(a)=y(b)=0,$ to have nontrivial solutions is that $\int_a^b |q(t)|dt > \frac{4}{b-a}$. There are several different proofs of this inequality since the original one by Lyapunov [1]. In the last few years independent works appeared generalizing Lyapunov's inequality for the *p*- Laplacian, by using Hö lder, Jenson or Cauchy-Schwarz inequalities. We refer reader to see references [2-9].

In 1949 Hahn [10] introduced the $q.\omega$ -difference operator $D_{q.\omega}$, say Hahn's difference operator. It has been applied successfully in construction of families of orthogonal polynomials as well as in approximation problem [11-13]. However, during 68 years, few authors have studied Hahn's quantum calculus. We refer reader to the monographs of Aldwoah [14][15], Bangerezako[16] and Artur M. C Brito da cru Z [17].

In this paper, we consider boundary value problem of following quaslinear $q.\omega$- difference equation

$$\begin{cases} -D_{q.\omega}\left(r(t)\left|D_{q.\omega}u(t)\right|^{p-2}(D_{q.\omega}u(t))\right)=f(tq+\omega)\left|u(tq+\omega)\right|^{p-2}u(tq+\omega), t\in(\omega_0,b), \\ u(\omega_0)=u(\mathrm{b})=0, u(t)\neq 0, t\in(\omega_0,b). \end{cases} \tag{1}$$

And

$$\begin{cases} -\mathrm{D}_{q.\omega}(r_1(t)\,|\,\mathrm{D}_{q.\omega}u(t)\,|^{p_1-2}(\mathrm{D}_{q.\omega}u(t)))=f_1(tq+\omega)\left|u(tq+\omega)\right|^{\alpha_1-2}u(tq+\omega)\left|v(tq+\omega)\right|^{\alpha_2}, t\in(\omega_0,b), \\ -\mathrm{D}_{q.\omega}(r_2(t))\,|\,\mathrm{D}_{q.\omega}v(t)\,|^{p_2-2}(\mathrm{D}_{q.\omega}v(t)))=f_2(tq+\omega)\left|v(tq+\omega)\right|^{\beta_2-2}v(tq+\omega)\left|u(tq+\omega)\right|^{\beta_1}, t\in(\omega_0,b), \\ u(\omega_0)=u(b)=v(\omega_0)=v(b)=0, u(t)\neq 0, v(t)\neq 0, t\in(\omega_0,b). \end{cases} \tag{2}$$

For the save of convenience, we give the following hypothesis (H_1) for (1) and hypothesis and (H_2) for (2):

(H_1) $r(t)$ and $f(t)$ are real-valued functions and $r(t) > 0$ for all $t \in R$, $1 < p, p_1 < \infty$, satisfy $\frac{1}{p_1} + \frac{1}{p} = 1$.

(H_2) $1 < p_1, p_2 < \infty, \alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ satisfy $\frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} = 1$ and $\frac{\beta_1}{p_1} + \frac{\beta_2}{p_2} = 1$.

We recall some concepts for $q.\omega$-difference operator.

Throughout this paper, we assume $q \in (0,1)$ and $\omega > 0$. We introduce the real number $\omega_0 := \frac{1}{1-q}$.

Let I be a real interval containing ω_0. For a function f defined on I, *Hahn's* difference operator $D_{q.\omega} f$ is given by

$$\left(D_{q.\omega} f\right)(x) = \frac{f(qx+\omega) - f(x)}{(q-1)x + \omega}, x \neq \omega_0, \left(D_{q.\omega} f\right)(\omega_0) = f'(\omega_0),$$

Definition 1 [17]. Let $a, b \in I$ and $a < b$. For $f : I \to R$ the $q.\omega$-integrals is defined by

$$\int_a^b f(x) d_{q.\omega} x = \int_{\omega_0}^b f(x) d_{q.\omega} x - \int_{\omega_0}^a f(x) d_{q.\omega} x,$$

$$\int_{\omega_0}^x f(t) d_{q.\omega} t = (x(1-q) - \omega) \sum_{n=0}^{\infty} q^n f(xq^n + \omega[n]_q),$$

Provided that the series converges at $x = a, x = b$. In that case, f is called $q.\omega$ integrable on $[a,b]$. We say that f is $q.\omega$ integrable over I if it is $q.\omega$ integrable on $[a,b]$ for all $a, b \in I$.

The $q.\omega$-difference analogue of Leibnitz rule is given by

$$D_{q.\omega} f(x) g(x) = g(qx+\omega) D_{q.\omega} f(x) + f(x) D_{q.\omega} g(x). \quad (3)$$

II. Lyapunov-Type inequalies for $q.\omega$-Symmetric Difference Equation (1)

In the section, we establish Lyapunov-type inequalies for $q.\omega$-symmetric difference equation (1).

Denote

$$\xi(t) = \left(\int_{\omega_0}^t r^{1-p_1}(s) \, \mathrm{d}_{q.\omega} s\right)^{\frac{1}{p_1-1}},$$

$$\eta(t)=(\int_t^b r^{1-p_1}(s)\,\mathrm{d}_{q.\omega}s)^{\frac{1}{p_1-1}}.$$

Theorem 2.1. Suppose that hypothesis (H_1) holds. If the boundary value problem (1) has a solution. Then one has the following inequality

$$\int_{\omega_0}^b \frac{\xi(\mathrm{q}t+\omega)\eta(\mathrm{q}t+\omega)}{\xi(\mathrm{q}t+\omega)+\eta(t\,\mathrm{q}t+\omega)} f^+(\mathrm{q}t+\omega)\,\mathrm{d}_{q.\omega}t \ge 1$$

where $f^+(t)=\max\{f(t),0\}$.

Proof. By (1) and (3), we have

$$\int_{\omega_0}^b f_1(\mathrm{q}t+\omega)\left|u(\mathrm{q}t+\omega)\right|^p d_{q.\omega}t$$

$$=-\int_{\omega_0}^b D_{q.\omega}\left(r(t)\left|D_{q.\omega}u(t)\right|^{p-2}\left(D_{q.\omega}u(t)\right)\right)u(tq+\omega)d_{q.\omega}t$$

$$=\int_{\omega_0}^b r(t)\left|D_{q.\omega}\,\mathrm{u}(t)\right|^{p-2}D_{q.\omega}u(t)D_{q.\omega}u(t)d_{q.\omega}t-\int_{\omega_0}^b D_{q.\omega}(u(t)\left((r(t)\left|D_{q.\omega}u(t)\right|^{p-2}D_{q.\omega}u(t))\right)d_{q.\omega}t$$

$$=\int_{\omega_0}^b r(t)\left|D_{q.\omega}(t)\right|^p d_{q.\omega}t \tag{4}$$

By the boundary condition of (1), we can get

$$\left|u(t)\right|^p=\left|\int_{\omega_0}^t D_{q.\omega}u(s)d_{q.\omega}s\right|^p \le (\int_{\omega_0}^t \left|D_{q.\omega}u(s)\right|d_{q.\omega}s)^p$$

$$=(\int_{\omega_0}^t r^{-\frac{1}{p}}(s)r^{\frac{1}{p}}(s)\left|D_{q.\omega}u(s)\right|d_{q.\omega}s)^p$$

$$\le(\int_{\omega_0}^t r^{1-p}(s)d_{q.\omega}s)^{\frac{1}{p_1-1}}\int_{\omega_0}^t r(s)\left|D_{q.\omega}u(s)\right|^p d_{q.\omega}s$$

$$=\xi(t)\int_{\omega_0}^t r(s)\left|D_{q.\omega}u(s)\right|^p d_{q.\omega}s$$

and

$$\left|u(t)\right|^p=\left|-\int_t^b D_{q.\omega}u(s)d_{q.\omega}s\right|^p \le \left[\int_t^b \left|D_{q.\omega}u(s)\right|d_{q.\omega}s\right]^p$$

$$=\left(\int_t^b r^{1-p}(s)d_{q.\omega}s\right)^{\frac{1}{p-1}}\left(\int_t^b r(s)\left|D_{q.\omega}u(s)\right|^p d_{q.\omega}s\right)\le\left(\int_t^b r^{1-p_1}(s)d_{q.\omega}s\right)^{\frac{1}{p-1}}\left(\int_t^b r(s)\left|D_{q.\omega}u(s)\right|^p d_{q.\omega}s\right)$$

$$=\eta(t)\int_t^b r(s)\left|D_{q.\omega}u(s)\right|^p d_{q.\omega}s.$$

Thus $\left|u(t)\right|^p \le \dfrac{\xi(t)\eta(t)}{\xi(t)+\eta(t)}\int_{\omega_0}^b r(s)\left|D_{q.\omega}u(s)\right|^p d_{q.\omega}s$

Therefore

$$\left|u\left(qt+\omega\right)\right|^{p}\le\frac{\xi\left(qt+\omega\right)\eta\left(qt+\omega\right)}{\xi\left(qt+\omega\right)+\eta\left(qt+\omega\right)}\int_{\omega_0}^{b}r\left(s\right)\left|D_{q.\omega}u\left(s\right)\right|^{p}d_{q.\omega}s.$$

By (4), we have

$$\int_{\omega_0}^{b}f^{+}\left(tq+\omega\right)\left|u\left(tq+\omega\right)\right|^{p}d_{q.\omega}t\le\int_{\omega_0}^{b}\frac{\xi\left(qt+\omega\right)\eta\left(qt+\omega\right)}{\xi\left(qt+\omega\right)+\eta\left(qt+\omega\right)}f^{+}\left(tq+\omega\right)d_{q.\omega}t\cdot\int_{\omega_0}^{b}r\left(s\right)\left|D_{q.\omega}u\left(s\right)\right|^{p}d_{q.\omega}s$$

$$=\int_{\omega_0}^{b}\frac{\xi\left(qt+\omega\right)\eta\left(qt+\omega\right)}{\xi\left(qt+\omega\right)+\eta\left(qt+\omega\right)}f^{+}\left(qt+\omega\right)d_{q.\omega}t\cdot\int_{\omega_0}^{b}f_1\left(qt+\omega\right)\left|u\left(qt+\omega\right)\right|^{p}d_{q.\omega}t.$$

$$\le\int_{\omega_0}^{b}\frac{\xi\left(qt+\omega\right)\eta\left(qt+\omega\right)}{\xi\left(qt+\omega\right)+\eta\left(qt+\omega\right)}f^{+}\left(qt+\omega\right)d_{q.\omega}t\cdot\int_{\omega_0}^{b}f^{+}\left(qt+\omega\right)\left|u\left(qt+\omega\right)\right|^{p}d_{q.\omega}t\,. \tag{5}$$

Next, we prove that $\int_{\omega_0}^{b}f^{+}\left(qt+\omega\right)\left|u\left(qt+\omega\right)\right|^{p}d_{q.\omega}t>0.$ (6)

If (6) is not true, then $\int_{\omega_0}^{b}f^{+}\left(qt+\omega\right)\left|u\left(qt+\omega\right)\right|^{p}d_{q.\omega}t=0.$ (7)

From (4) and (7), we have

$$0\le\int_{\omega_0}^{t}r\left(s\right)\left|D_{q.\omega}u\left(s\right)\right|^{p}d_{q.\omega}s\le\int_{\omega_0}^{b}r\left(s\right)\left|D_{q.\omega}u\left(s\right)\right|^{p}d_{q.\omega}s$$
$$=\int_{\omega_0}^{b}f\left(qt+\omega\right)\left|u\left(qt+\omega\right)\right|^{p}d_{q.\omega}t\le\int_{\omega_0}^{b}f\left(qt+\omega\right)\left|u\left(qt+\omega\right)\right|^{p}d_{q.\omega}t=0.$$

It follows $D_{q.w}u\left(q^{n}t+\omega\right)\equiv 0, n=0,1,...,$, we obtain that $u\left(t\right)\equiv 0$, for $t\in(\omega_0,b)$ which contradicts the condition (2). Therefore, from (5), we may see that theorem 2.1 holds.

Note that $(\frac{\xi+\eta}{2})^2\ge\xi\eta$, one has following corollary 2.1.

Corollary 2.1. Suppose that hypothesis (H_1) is satisfied. If (1) has a solution $u\left(t\right)$. Then one has the following inequality.

$$\int_{\omega_0}^{b}(\xi(\mathrm{q}t+\omega)\eta(\mathrm{q}t+\omega))^{\frac{1}{2}}f^{+}(\mathrm{q}t+\omega)d_{q.\omega}t\ge 2$$

Ⅲ Lyapunov-type Inequalities for $q.\omega$ -difference system(2)

Denote

$$\xi_i(t)=(\int_{\omega_0}^{t}r_i^{1-p_i}(s)d_{q.\omega}s)^{\frac{1}{p_i-1}},i=1,2 \tag{8}$$

$$\eta_i(t)=(\int_t^b r_i^{1-p_i}(s)d_{q.\omega}s)^{\frac{1}{p_i-1}}, i=1,2 \tag{9}$$

Theorem 3.1. Suppose that hypothesis (H_2) is satisfied. If system (2) has a solution $\left(u(t),v(t)\right)$. Then one has the following inequality:

$$M_{11}^{\alpha_1\beta_1/p_1^2}M_{12}^{\beta_1\alpha_2/p_1p_2}M_{21}^{\beta_1\alpha_2/p_1p_2}M_{22}^{\alpha_2\beta_2/p_2^2}\geq 1$$

where $M_{ij}=\int_0^1 \frac{\xi_i(\mathrm{q}t+\omega)\eta_i(\mathrm{q}t+\omega)}{\xi_i(\mathrm{q}t+\omega)+\eta_i(\mathrm{q}t+\omega)} f_j^+(\mathrm{q}t+\omega)d_{q.\omega}t$ i, j=1, 2

where $f_j^+(t)=\max\left\{f_i(t),0\right\}$, for i = 1, 2

Proof. Similar to (4), we have

$$\int_{\omega_0}^b r_1(s)\,|D_{q.\omega}u(s)|^{p_1}\,d_{q.\omega}s=\int_{\omega_0}^b f_1(\mathrm{q}s+\omega)\,|u(\mathrm{q}s+\omega)|^{\alpha_1}\,|v(\mathrm{q}s+\omega)|^{\alpha_2}\,d_{q.\omega}s \tag{10}$$

$$\int_{\omega_0}^b r_2(s)\,|D_{q.\omega}v(s)|^{p_2}\,d_{q.\omega}s=\int_{\omega_0}^b f_2(\mathrm{q}s+\omega)\,|v(\mathrm{q}s+\omega)|^{\beta_2}|u(\mathrm{q}s+\omega)|^{\beta_1}\,d_{q.\omega}s \tag{11}$$

$$|u(t)|^{p_1}\leq \frac{\xi_1(t)\eta_1(t)}{\xi_1(t)+\eta_1(t)}\int_{\omega_0}^b r_1(s)\,|D_{q.\omega}u(s)|^{p_1}\,d_{q.\omega}s \tag{12}$$

$$\int_{\omega_0}^b f_1(\mathrm{q}t+\omega)\,|u(\mathrm{q}t+\omega)|^{p_1}d_{q.\omega}t\leq M_{11}\int_{\omega_0}^b r_1(s)\,|D_{q.\omega}u(s)|^{p_1}\,d_{q.\omega}s$$

$$=M_{11}\int_{\omega_0}^b f_1(\mathrm{q}s+\omega)\,|u(\mathrm{q}s+\omega)|^{\alpha_1}\,|v(\mathrm{q}s+\omega)|^{\alpha_2}\,d_{q.\omega}s$$

$$\leq M_{11}\int_{\omega_0}^b f_1^+(\mathrm{q}s+\omega)\,|u(\mathrm{q}s+\omega)|^{\alpha_1}\,|v(\mathrm{q}s+\omega)|^{\alpha_2}\,d_{q.\omega}s$$

$$\leq M_{11}(\int_{\omega_0}^b f_1(\mathrm{q}s+\omega)\left|u(\mathrm{q}s+\omega)\right|^{p_1}d_{q.\omega}s)^{\frac{\alpha_1}{p_1}}(\int_{\omega_0}^b f_1^+(\mathrm{q}s+\omega)\left|v(\mathrm{q}s+\omega)\right|^{p_2}d_{q.\omega}s)^{\frac{\alpha_2}{p_2}} \tag{13}$$

$$\begin{aligned}\int_{\omega_0}^b f_2^+(\mathrm{q}t+\omega)\left|u(\mathrm{q}t+\omega)\right|^{p_1}d_{q.\omega}t &\leq M_{12}(\int_{\omega_0}^b f_1^+(\mathrm{q}t+\omega)\left|u(\mathrm{q}t+\omega)\right|^{p_1}d_{q.\omega}t)^{\frac{\alpha_1}{p_1}}\\ &\times(\int_{\omega_0}^b f_1^+(\mathrm{q}t+\omega)\left|v(\mathrm{q}t+\omega)\right|^{p_2}d_{q.\omega}t)^{\frac{\alpha_2}{p_2}}\end{aligned} \tag{14}$$

Similar to the proof of (12), from (10) (11), we have

$$\left|v(s)\right|^{p_2}\leq\frac{\xi_2(t)\eta_2(t)}{\xi_2(t)+\eta_2(t)}\int_{\omega_0}^b r_2(\mathrm{s})\left|D_q v(s)\right|^{p_2}d_{q.\omega}s$$

It follows from above form and the Hö lder inequality that

$$\int_{\omega_0}^{b} f_1^+(qt+\omega)\left|v(qt+\omega)\right|^{p_2} d_{q.\omega}t \le \int_{\omega_0}^{b} \frac{\xi_2(qt+\omega)\eta_2(qt+\omega)}{\xi_2(qt+\omega)+\eta_2(qt+\omega)} f_1^+(qt+\omega)\, d_{q.\omega}t \int_{\omega_0}^{b} r_2(s)\left|D_q v(s)\right|^{p_2} d_{q.\omega}s$$

$$\le M_{12}\left(\int_{\omega_0}^{b} f_2^+(qt+\omega)\left|u(qt+\omega)\right|^{\beta_1}\left|v(qt+\omega)\right|^{\beta_2} d_{q.\omega}t\right)$$

$$\le M_{12}\left(\int_{\omega_0}^{b} f_2^+(qt+\omega)\left|u(qt+\omega)\right|^{p_1} d_{q.\omega}t\right)^{\frac{\beta_1}{p_1}}\left(\int_{\omega_0}^{b} f_2^+(qt+\omega)\left|v(qt+\omega)\right|^{p_2} d_{q.\omega}t\right)^{\frac{\beta_2}{p_2}} \tag{15}$$

$$\int_{\omega_0}^{b} f_2^+(qt+\omega)\left|v(qt+\omega)\right|^{p_2} d_{q.\omega}t \le M_{12}\left(\int_{\omega_0}^{b} f_2^+(qt+\omega)\left|u(qt+\omega)\right|^{p_1} d_{q.\omega}t\right)^{\frac{\beta_1}{p_1}} \times\left(\int_{\omega_0}^{b} f_2^+(qt+\omega)\left|v(qt+\omega)\right|^{p_2} d_{q.\omega}t\right)^{\frac{\beta_2}{p_2}} \tag{16}$$

Similar to (5), we can get

$$\int_{\omega_0}^{b} f_1^+(qt+\omega)\left|u(qt+\omega)\right|^{p_1} d_{q.\omega}t > 0,\ \int_{\omega_0}^{b} f_2^+(qt+\omega)\left|u(qt+\omega)\right|^{p_1} d_{q.\omega}t > 0,$$

$$\int_{\omega}^{b} f_1^+(qt+\omega)\left|v(qt+\omega)\right|^{p_1} d_{q.\omega}t > 0,\ \int_{\omega_0}^{b} f_2^+(t)\left|v(t)\right|^{p_2} d_{q.\omega}t > 0. \tag{17}$$

From (14)-(17), we have

$$M_{11}^{\frac{\alpha_1\beta_1}{p_1^2}} M_{12}^{\frac{\beta_1\alpha_2}{p_1p_2}} M_{21}^{\frac{\beta_1\alpha_2}{p_1p_2}} M_{22}^{\frac{\alpha_2\beta_2}{p_2^2}} \ge 1$$

Corollary3.1. Suppose that hypothesis (H_2) are satisfied. If (2) has a solution $\left(u(t),v(t)\right)$. Then

$$2 \le \left(\int_{\omega_0}^{b} (\xi_1(qt+\omega)\eta_1(qt+\omega))^{\frac{1}{2}} f_1^+(qt+\omega)\, d_{q.\omega}t\right)^{\frac{\beta_1\alpha_2}{p_1^2}} \left(\int_{\omega_0}^{b} (\xi_2(qt+\omega)\eta_2(qt+\omega))^{\frac{1}{2}} f_2^+(qt+\omega)\, d_{q.\omega}t\right)^{\frac{\beta_1\alpha_2}{p_1p_2}}$$

$$\times\left(\int_{\omega_0}^{b} (\xi_1(qt+\omega)\eta_1(qt+\omega))^{\frac{1}{2}} f_1^+(qt+\omega)\, d_{q.\omega}t\right)^{\frac{\beta_1\alpha_2}{p_1p_2}} \left(\int_{\omega_0}^{b} (\xi_2(qt+\omega)\eta_2(qt+\omega))^{\frac{1}{2}} f_2^+(qt+\omega)\, d_{q.\omega}(t)\right)^{\frac{\beta_1\alpha_2}{p_2^2}}$$

Next, we consider the quasilinear $q.\omega$ -symmetric system involving the ($P_1, P_2, \cdots, P_m$)-Laplacian:

$$\begin{cases} -\tilde{D}_q(r_1(t)\left|D_q u_1(t)\right|^{p_1-2}(D_q u_1(t))) = f_1(qt+\omega)\left|u_1(qt+\omega)\right|^{\alpha_1-2}\left|u_2(qt+\omega)\right|^{\alpha_2}\cdots\left|u_m(qt+\omega)\right|^{\alpha_m} u_1(qt+\omega) \\ -\tilde{D}_q(r_2(t)\left|D_q u_2(t)\right|^{p_2-2}(D_q u_2(t))) = f_2(qt+\omega)\left|u_1(qt+\omega)\right|^{\alpha_1}\left|u_2(qt+\omega)\right|^{\alpha_2-2}\cdots\left|u_m(qt+\omega)\right|^{\alpha_m} u_2(qt+\omega) \\ \vdots \\ -\tilde{D}_q(r_m(t)\left|D_q u_m(t)\right|^{p_m-2}(D_q u_m(t))) = f_m(qt+\omega)\left|u_1(qt+\omega)\right|^{\alpha_1}\left|u_2(qt+\omega)\right|^{\alpha_2}\cdots\left|u_m(qt+\omega)\right|^{\alpha_m-2} u_m(qt+\omega) \end{cases} \tag{18}$$

with boundary value conditions:

$$u_i(\omega_0) = u_i(b) = 0, u_i(t) \ne 0, t \in (\omega_0, b), i = 1, 2, \cdots, m. \tag{19}$$

We give the following hypothesis (H_3).

(H_3) $r_i(t)$ and $f_i(t)$ are real-valued functions and $r_i(t) \geq 0$ for $i = 1, 2, ..., m$.

Furthermore, $1 < p_i < \infty$ and $\alpha_i > 0$ satisfy $\sum_{i=1}^{m} (\frac{\alpha_i}{p_i}) = 1$.

Denote $$\xi_i(t) = (\int_{\omega_0}^{t} r_i^{1-p_i}(\mathrm{s})\, d_{q.\omega}\, s)^{\frac{1}{p_i - 1}}$$

$$\eta_i(t) = (\int_{t}^{b} r_i^{1-p_i}(\mathrm{s})\, d_{q.\omega} s)^{\frac{1}{p_i - 1}} \tag{20}$$

Theorem 3.2. Suppose that hypothesis (H_3) is satisfied. If system (18) has a solution $(u_1(t), u_2(t), ...u_m(t))$ satisfying the boundary condition (19), then one has the following inequality:

$$\coprod_{i=1}^{m} \prod_{j=1}^{m} (\int_{\omega_0}^{b} \frac{\xi_i(\mathrm{q}\tau + \omega)\eta_i(\mathrm{q}\tau + \omega)}{\xi_i(\mathrm{q}\tau + \omega) + \eta_i(\mathrm{q}\tau + \omega)} f_j^{+}(\mathrm{q}\tau + \omega)\, d_{q.\omega}\, \tau)^{\frac{\alpha_i \alpha_j}{p_i p_j}} \geq 1 \tag{21}$$

Proof. By (18) (H_3) and (19), we have

$$\int_{\omega_0}^{b} r_i(t) \left|D_q u_i(t)\right|^{p_i} d_{q.\omega}\, t \leq q \int_{\omega_0}^{b} f_i(\mathrm{q}t + \omega) \prod_{k=1}^{m} \left|u_k(\mathrm{q}t + \omega)\right|^{\alpha_k} d_{q.\omega}\, t, i = 1, 2, ..., m$$

It follows from (20) and the Hö lder inequality that

$$\left|u_i(t)\right|^{p_i} \leq \frac{\xi_i(\tau)\eta_i(\tau)}{\xi_i(\tau) + \eta_i(\tau)} \int_{\omega_0}^{b} r_i(t) \left|D_q u_i(t)\right|^{p_i} d_{q.\omega} t$$

$$\int_{\omega_0}^{b} f_j^{+}(\mathrm{q}t + \omega) \left|u_i(\mathrm{q}t + \omega)\right|^{p_2} d_{q.\omega}\, t \leq M_{ij} \prod_{k=1}^{m} (\int_{\omega_0}^{b} f_i^{+}(\mathrm{q}t + \omega) \left|u_k(\mathrm{q}t + \omega)\right|^{p_k} d_{q.\omega}\, t)^{\frac{\alpha_k}{p_k}}$$

where

$$M_{ij} = \int_{\omega}^{b} \frac{\xi_i(\mathrm{q}t + \omega)\eta_i(\mathrm{q}t + \omega)}{\xi_i(\mathrm{q}t + \omega) + \eta_i(\mathrm{q}t + \omega)} f_j^{+}(\mathrm{q}t + \omega)\, d_{q.\omega}\, t, i = 1, 2, ..., m \tag{22}$$

Similar to the proof of the (17), we get

$$\int_{\omega_0}^{b} f_i(\mathrm{q}t + \omega) \left|u_k(\mathrm{q}t + \omega)\right|^{p_k} d_{q.\omega} t > 0, i, k = 1, 2, ..., m$$

Therefore

$$\prod_{i=1}^{m}\prod_{j=1}^{m}(M_{ij})^{\frac{\alpha_i\alpha_j}{p_i p_j}} \geq 1 \tag{23}$$

It follows from (22) and (23) that (21) holds.

Corollary3.2.Suppose that hypothesis (H_3) are satisfied, if the system(18)has a solution $(u_1(t),u_2(t),...u_m(t))$ satisfies(19), then one has the following inequality:

$$\prod_{i=1}^{m}\prod_{j=1}^{m}\left(\int_{\omega_0}^{b}(\xi_i(\mathrm{q}t+\omega)\eta_i(\mathrm{q}t+\omega))^{\frac{1}{2}}\,\mathrm{f}_j^{+}(\mathrm{q}t+\omega)\,d_{q.\omega}t\right)^{\frac{\alpha_i\alpha_i}{p_i p_j}} \geq 2$$

References

[1]. A.Lyapunov,Problem General de la Stabilite du Mouvement,in:Ann.Math.Studies,vol.17,Prin-ceton Univ.Press, 1949, Reprinted from Ann. Fac.Sci.Toulouse,9(1907)207-474, Translation of the original paper published in Comm.Soc.Math. Kharkow. 1892.

[2]. R.P. Agarwal, C.-F.Lee, C.-C.Yeh, C.-H.Hong, Lyapunov and Writinger inequalities, Appl. Math. Lett.17(2004)847-853.

[3]. D.Cakmak, Lyapunov-type integral inequalities for certain higher order differential equations, Appl. Math.Comp 216(2010)368-373.

[4]. A.Canada, J.A.Montero, S.Villegas,Lyapunov inequalities for partial differential equations, J. Funct. Anal. 237(2006)176-193.

[5]. M.K.Kwong, On Lyapunov inequality for disfocality, J. Math.Anal.Appl.83(1981)486-494.

[6]. B.G.Pachpatte, Lyapunov type integral inequalities for certain differential equations, Georgian Math. J.4(1997)139-148.

[7]. J.P.Pinasco, Lower bounds for eigenvalues of the one-dimensional *p*-Laplacian, Abstr. Appl. Anal. 2004(2004)147-153.

[8]. J.P.Pinasco, Comparison of eigenvalues for the *p*-Laplacian with integral inequalities, Appl. Math. Comput. 182(2006)1399-1404.

[9]. X.Yang, K.Lo, Lyapunov-type inequality for a class of even-order differential equalitions,Appl. Math. Comput.215(2010)3884-3890.

[10]. W. Hahn, Über orthogonalpolynome, die q-Differenzengleichungen genugen, Math.Nachr. 2(1949)4-34.

[11]. R.Alvarez-Nodarse, On characterizations of classical polynomials, J. Comput.Appl.Math.196(1)(2006) 320-337.

[12]. A.Dobrogowska, AOdzijewica, Second oder q-differen equations solvable by factorization method.J.Comput.Appl.Math.193(1)(2006)319-346.

[13]. J.Petroniho,Generic formulas for the values at the singular points of some special monic classical H-orthogonal polynomial, J. Comput,Appl Math. 205(1)(2007)314-324.

[14]. K.A.Aldwoah, Generalized time scales and associated difference equations, Ph.D.Thesis,Cairo University,2009.

[15]. K.A.Aldwoah,A.E.Hamza,Difference time scales, Int. J. Math.Stat. 9(A11)(2011)106-125.

[16]. G.Bangerezako, Variational q-calculus. J.Math.Anal. Appl. 289(2)(2004)650-665.

[17]. Artur M.C. Brito da Cruz, Natalia Martins, Delfim F. M. Torres, Higher-order Hahn's quantum variational calculus. Nonlinear Analysis. 75(2012)1147-1157.