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Stochastization of the Exponential form Learning Curve by ItÔ Processing

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Abstract: Cost forecasting is a vital process that determines probable future costs, the value which when deducted from expected revenues projects profit in the income equation. For especially labour intensive business concerns whose labour costs form a substantial proportion of total cost, it is prudent to adopt a convenient forecasting method that most accurately estimates expected labour costs. The exponential form learning curve is deterministic for which reason the cost range for a set of inputs can only be obtained by the labourious sensitivity analysis. This paper formulated a stochastic cost forecasting method by aid of an algorithm in R – statistical application. First, using hypothetical data, it converted deterministic marginal and cumulative average exponential form learning curves into their equivalent stochastic forms by Ito processing – a specialized form of geometric Brownian motion model. Secondly, it constructed a stochastic learning curve from secondary data providing drift, output volatility and simulation runs as variables which capture more information hence more robust than the ordinary deterministic exponential form learning curve and include confidence intervals that assist all levels of efficiencies of labour. The correlation between deterministic and stochastic learning curve forms was 100% for hypothetical data in the regression sense. For secondary data, the stochastic learning curve form proved a better fit; recording a coefficient of determination (R^2) of 0.96826 compared to the Bayesian form where R^2 of 0.93101 had been posted. It is recommended that the stochastic forecasting be adopted by reason of robustness and flexibility.

Keywords: Ito process, learning curve, deterministic, stochastic, geometric Brownian motion model, simulation

Introduction

After Hermann Ebbinghaus first described the learning curve in the field of psychology in 1885 without using the term, Theodore Paul Wright operationalized the concept within a manufacturing setting by publishing the first formal paper in 1936 (Harvard Business review, 2014). Later in 1979 Yelle coined the term learning curve which is used to date. The idea is that time spent in manufacturing decreases at more or less a constant rate every time the volume is doubled. Companies all over the world have enhanced their business models by working around the learning curve. The learning curve theory has found tremendous use in manufacturing, incentive development and development of training programmes in industry. Scholars and industrial practitioners are interested in the aspect of time reduction (Fierenzo, 2004) necessary to optimize productivity.

Time reduction is viewed through the forecasting lens. This predictive element is invaluable, especially in the light of setting up a new manufacturing plant. Accurate predictive algorithms which consider all the factors affecting a comprehensive manufacturing system need to be developed (Naim, 1993). Modeling process then follows where subsisting relationships between variables are explored and tested to and agreeable level of accuracy. These relationships are referred to as “composition laws” which can provide a forecasting tool of the behavior of complex manufacturing systems (Franceschini and Galetto, 2003). Learning is both associated with non-human and human components of a system; where the human components exhibit greater learning which affects the overall efficiency of a system more materially. Parts of a new machine need to get used to operation by gradual increase of runs, possibly to even out grease in all the parts, thereby increasing efficiency. However, this efficiency increases only to a limited extent. Human efficiency on the other hand is quite elastic, extending even tenfold (Jovanovic and Nyarko, 1995). This gives credence to experience; no wonder the other name for learning curves is experience curves. But human productivity is not just influenced by experience. A myriad other factors including momentary emotional disposition, sickness among others, come into play to affect a deterministic productivity relationship. Most learning curve models are deterministic. Whereas related deviations from empirical observations can be addressed by sensitivity analysis, this is likely to be quite labourious and time consuming. It is therefore important to model a human related learning curve stochastically to accommodate obvious human related deviations. Learning curves can be modeled by all manner of mathematical functions. The basic exponential form can be represented as $y = ax^b$, where y represents either marginal or cumulative average production time, a is the time taken to produce the first unit, x is the number of units and b is the learning coefficient. When logarithms are taken on both sides of the equation, a straight line learning curve obtains (Bendrey, Hussey & West, 2003), which is a fact.

Types of Learning Curves

Learning curves are quite diverse bivariate functions which can be plotted on a Cartesian plane to exhibit increasing or decreasing monotone functions. By interpolation, variance analysis in production settings can be worked out to secure requisite administrative action and by extrapolation, necessary productivity forecasting can be done. Types of exponential curves include: linear functions with negative gradients, reciprocal functions, portions of polynomial functions for instance quadratic and cubic functions, and exponential functions (Bendrey, Hussey & West, 2003). Ordinarily, polynomials apply only within limited portions and are therefore not going to be given particular attention. The linear function is symbolically denoted as $y = h - kx$ where x and y are variables, h and k are real valued non-negative constants. In the learning context h, normally being the y intercept there exists a non-negative time duration within which zero units are manufactured. This is one of the disadvantages of the linear model. Only the exponential form learning curve was of interest.

Statement of the Problem and Objectives

Scholars and decision makers in industry agree that different people post different learning curves by reason of different capabilities. Few, however, of these curves incorporate the stochastic element of human learning. Besides, human learning is Markovian. That is, new learning is only dependent on the immediate past learning. In this context, envisioned is a scenario whereby a group of employees are assumed not to have interacted with the process in question previously, and that all learning takes place purely from their interaction with the process. Further, it is assumed that the process comprises related sub-operations all of which are new to the employee. This aspect is normally not captured in most models (Jovanovic and Nyarko, 1995). The decision maker is in most cases not in a position to assign confidence levels at the projected learning or productivity levels. The aim is therefore to transform the deterministic exponential form learning curve into a stochastic learning curve; utilizing an Ito process. An Ito process is a special form of a geometric Brownian motion stochastic differential equation in which change in the estimated quantity depends both on the immediate previous quantity (at the previous time) and time interval as shown by equation 4. Moreover, the change equation has two terms: the deterministic and the stochastic Gaussian White noise term (first and second terms on the right hand side of equation 4 respectively). Change in the estimated quantity depends on the two terms each of which is a function of the immediate last magnitude of the quantity. To address the determinism anomaly, the paper envisages achieving three objectives namely: to transform a marginal learning curve into a stochastic learning curve; secondly, to transform a cumulative average learning curve into its equivalent stochastic form and lastly to generate a stochastic learning curve from real data from which to forecast subsequent learning and productivity including the interval estimate at 95% confidence level.

Methodology

Consider an 80% exponential learning curve represented by the equation:

$$y = ax^b \quad \text{……………………….. (1)}$$

For cumulative average model interpretation, y equals the cumulative average time taken to produce x units; while for the marginal production learning curve, y equals the time taken to produce the x^{th} unit. For both models, a is the time taken to produce the first unit and b is the learning coefficient. This is a deterministic model since fixing the values of a, x and b yields a unique solution, also known as a point estimate statistically.

$$b = \frac{\log\left(1 - \textit{proportionate decrease}\right)}{\log 2} \quad \text{………………………. (2)}$$

Table 1: Deterministic cumulative average and marginal unit production time learning curves

cumulative pdn (units) (a)	cumulative time for avg. (hours) (b)	cumulative time for marginal (hours) (c)	cumulative average time per unit (hours) (d)	Time taken for marginal unit (hours) (e)
1	50	50	50	50
2	80	90	40 (50x80%)	40 (50x80%)
4	128	157.1	32 (50x80%x80%)	32 (50x80%x80%)

8	204.8	267.28	25.6 (50x80%x80%x80%)	25.6 (50x80%x80%x80%)

Source: Management Accounting (Lucey,2003)

Column (a) shows the doubling process as production increases, while (b) cumulative average time which decreases with increase in production. Column (c) illustrates cumulative time for marginal units produced while (d) and (e) derive unit production time for the two methods.

This deterministic function was stochastized using a special process of geometric Brownian motion known as the Ito process (Hull, Treepongkaruna, Colwell, Heaney & Pitt, 2013) depicted by equation (3), using production rate rather than time taken as shown in table 2.

$$R_{t+1} = R_t + \Delta R_t \quad \text{……………………………… (3)}$$

$$\Delta R_t = \mu R_t \Delta t + \sigma R_t \varepsilon \sqrt{\Delta t} \quad \text{……………………………… (4)}$$

$$R_{t+1} = R_t + \mu R_t \Delta t + \sigma R_t \varepsilon \sqrt{\Delta t} \quad \text{……………………………… (5),}$$

where R_t is the production rate at time t, ΔR_t is the total change in production rate at time t, Δt is change in time, μ, σ, and ϵ are the drift (average rate of change of R), standard deviation of production of the workforce and a standard normal variable N (0, 1). The second therm in the right hand side of equation (4) is a Gaussian white noise term. The drift and standard deviation are assumed to remain constant during the process. An algorithm was constructed in R statistical software that utilizes equation (5) to describe the 80% learning curve using the deterministic function only to an acceptable correlation level of above 0.95. This was achieved by setting the Gaussian white noise term to zero then simulating at 40,000 runs considered sufficient for convergence to occur. The rate of production at time t devoid of volatility was obtained by deducting the Gaussian white noise term from equation (5). The difference between simulated Rt and simulated $R_{t-1} + \mu R_{t-1}\Delta t$ gives the value of the Gaussian white noise term at convergence. Note that R_{t-1} contains the white noise term but not $\mu R_{t-1}\Delta t$. Yet, from statistics, margin of error is inversely proportional to the square root of sample size as in equation (6).

$$n = \frac{z^2\sigma^2}{ME^2}$$ .... (6), where ME = margin of error; Z = standard normal statistic; and σ = standard deviation from the workforce. Therefore the margin of error restated from equation (5) and (6) results into:

$$ME = \sigma R_t \varepsilon \sqrt{\Delta t} = \frac{z\sigma}{\sqrt{n}} \quad \text{……………… (7)}$$

From equations and seven, it is clear that margin of error is unique to a specific production rate. The Gaussian white noise term can either be positive or negative occasioned by the normal random variable; since time, rate of production and standard deviation cannot be negative. Since sample size equals the number of simulation runs but the number of workers in question may be much less say 25 workers, the proportion of margin of error increases as shown in equation (8).

$$ME_t = \frac{\sqrt{40,000}}{\sqrt{25}}\left(\sigma R_t \varepsilon \sqrt{\Delta t}\right) \quad \text{………………………… (8)}$$

This margin of error may be substantial or negligible (almost a point estimate) depending on the size of the numbers involved. The worked out margins are included in table 2 for a constant standard deviation of 6.846.

Table 2: Deterministic and stochastically simulated production rates including margin of errors

Deterministic 80% cumulative/ marginal unit(s) (a)	Production time in hours (b)	Deterministic 80% cumulative/ marginal unit pdn rate (a)/(b) [units per hour]	Stochastic (Ito) cumulative/ marginal unit pdn rate (mean) (d)	Margin of error for stochastic function for 40,000 runs	Margin of error for stochastic function for 25runs (workers)
1	50	0.02	1	0	0
1.3872	45	0.0308	1.264028	± 0.002261072	± 0.09044288
2	40	0.05	1.807714	± 0.005184934	± 0.20739736

3.028	35	0.0865	2.729825	± 0.003409002	± 0.13636008
4.888	30	0.1629	4.456573	± 0.008577792	± 0.34311168
8.612	25	0.3445	7.952922	± 0.01285643	± 0.5142572
17.222	20	0.8611	16.32527	± 0.1216179	± 4.864716
42.09	15	2.806	42.02652	± 0.07879004	± 3.1516016
148.3	10	14.83	155.9021	± 2.070458	± 82.81832
1277	5	255.4	1224.073	± 8.031676	± 321.26704

This functions are not linear but Pearson's r has been used to approximate the accuracy of the model at about 100% accurate as shown in the correlation table 3. This suggests that the Ito function can almost exactly be used instead of the exponential curve under examination as illustrated by figure 1.

Table 3: Correlation deterministic with stochastic (Ito) processing

Correlations: Deterministic vs. Ito processing

		deterministic	Stochastic (Ito)
deterministic	Pearson Correlation	1	1.000**
	Sig. (2-tailed)		.000
	N	10	10
stochastic	Pearson Correlation	1.000**	1
	Sig. (2-tailed)	.000	
	N	10	10

**. Correlation is significant at the 0.01 level (2-tailed). $\mathbf{R^2 = 1}$

The first and second columns of table 2 show units produced in time intervals of 50 hours, 45 hours, 40 hours till 5 hours. Moreover, column (d) shows Ito processed productivity by adjusting the drift (average rate of increase in productivity). On average assuming the firm in question operates a 50 hour week, a worker is only able to produce one unit during the first week. For this reason, the annual standard deviation chosen later for purposes of introducing stochasticity will be subject to weekly intervals (1/52 weeks in a year) giving a time interval of 0.01923. The square root of this interval is 0.1387.

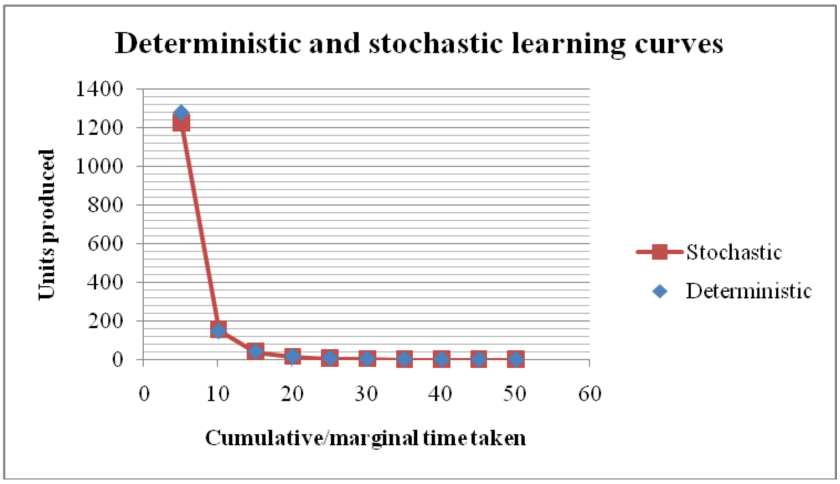

Figure 1: Deterministic and stochastic learning curves of the same process

In terms of cummulative average model, column (a) shows the number of units produced in the time shown in column (b) corresponding to every value in column (a). Effectively, the production rate per hour is obtained by dividing (a) by (b) resulting in column (c). The stochastic model means that 1.264028 ± 0.09044288 units will be produced in 45 hours in case of 25 shop floor workers and 1.264028 ± 0.002261072 in case of 40,000 shop floor workers. Notably, the margin of error for 25 workers is higher than that of 40, 000 workers since margin of error is inversely proportional to the square root of sample size as can be inferred from equation

(6). For the marginal unit model, column (d) may be interpreted to mean that 1.264028th unit plus or minus the margin of error will be produced in 45 hours. Expected production time ranges were worked out in table 4.

Table 4: Lower and upper limits of production time by Ito processing for a workforce of 25

Deterministic 80% cumulative/ marginal unit(s) (a)	Production time (hrs) (b)	Stochastic (Ito processed) production rates (c)	Margin of error for 25 workers	Lower limit of production (units)	Upper limit of production (units)	Lower limit of production time per unit (hrs)	Upper limit of production time per unit (hrs)
1	50	1	0	1	1	50	50
1.3872	45	1.26403	0.09044	1.17359	1.35447	33.22331	38.34404
2	40	1.80771	0.20740	1.60032	2.01511	19.85002	24.99505
3.028	35	2.72983	0.13636	2.59346	2.86619	12.21135	13.49546
4.888	30	4.45657	0.34311	4.11346	4.79968	6.25041	7.29313
8.612	25	7.95292	0.51426	7.43866	8.46718	2.95258	3.36082
17.222	**20**	**16.32527**	**4.86472**	**11.46055**	**21.18999**	**0.94384**	**1.74512**
42.09	15	42.02652	3.15160	38.87492	45.17812	0.33202	0.38585
148.3	10	155.90210	82.81832	73.08378	238.72042	0.04189	0.13683
1277	5	1224.07300	321.2670	902.80596	1545.34004	0.00324	0.00554

In the cumlative average sense, unit production time reduces progressively to a point where 17.222 units are produced in 20 hours (in bold) giving and average of 0.8611 units per hour (17.222/20) using the deterministic model. The stochastic model (Ito process) estimates this production at 16.32527. The advantage with Ito processing is that it avails a margin of error of 4.86472 (for this level of production), yielding upper and lower production limits of 21.18999 and 11.46055 respectively for a sample size of 25 workers. Dividing production time by each of the lower and upper production limits gives the upper and lower production time per unit respectively of 1.74512 and 0.94384. All other rows of the table may be interpreted similarly.

It becomes appropriate to introduce confidence limits at this point. Recall that the margin of error generated utilized a standard deviation arising from the group of workers under analysis. A fuether extension may be made in relation to equation (6) to work out the confidence intervals for z = 1.96, that is, for a 95% confidence level. This working can utilize equation (7) directly thus:

$ME = z\sigma/\sqrt{n} = 1.96x6.846/5 = 2.6836$

This kind of a margin of error will only be covered by production times of 20 hours and below (table 4). Higher production times featuring early in the production process are so unpredictable, given that the group is new to the task and probably unknown to the supervisors. It is during the first trial, that is, first unit production that the group's standard deviation is calculated. Many errors are expected at this learning stage therefore it is difficult to be sure of any production level. However, with increasing trials, the group bonds and consistency in production is established. Given that t = (x – mean)/standard deviation, working out z for row 3 in table 4, gives 0.20739736/6.846 = 0.0303 = t. This results in a slightly higher probability of 0.012. Comparing this to row 8, t = 3.1516016/6.846 = 0.4604. This yieds a substantial probability of slightly more than 0.1772. However, row 9 with margin of error of 82.81832 generates a t value of 82.81832/6.846 = 12.097. At this level of t, the range includes almost 100% of all possible vales. It means the production manager is sure that all possible productivity levels have been covered. In fact had the sample size been just above 30, the production manager would be 99% confident that 2.58 = (x – 155.9021)/6.846 that implies the margin of error would be within 17.663 units.

Ito processing of flight control learning emprical data

A similar process is followed in modeling actual data. Secondary but empirical data was taken from the work of Javanovic and Nyarko (1995) on 115 flight control trainees on 17 experiments as indicated in the first two columns of table 5. Ito processing was done on the data to arrive at the Ito success rate as shown in the same table. Unlike in Bayesian modeling method used by Javanovic and Nyarko where number of trainees is not important, Ito processing presumes that the 115 trainees have been sampled from a large population assumed to be normal by invoking the central limit theorem. It also assumes that the sample standard deviation of the trainees remains constant throughout the 17 experiments. Monte Carlo simulation was performed according to equation (5) to convergence of 40,000 runs. The margin of error was determined by deducting the immediate previous success rate and its deterministic drift term from the current success rate as shown in equation (9).

$$\sigma R_t \varepsilon \sqrt{\Delta t} = R_{t+1} - R_t - \mu R_t \Delta t \quad \text{.................................. (9)}$$

This margin of error is then proportionated for 115 trainees according to the subsisting inverse relationship to the square root of number of runs (trainees) to arrive at an expanded margin of error in table 5 for the group's standard deviation assumed to be 0.21 as indicated in figure 2.

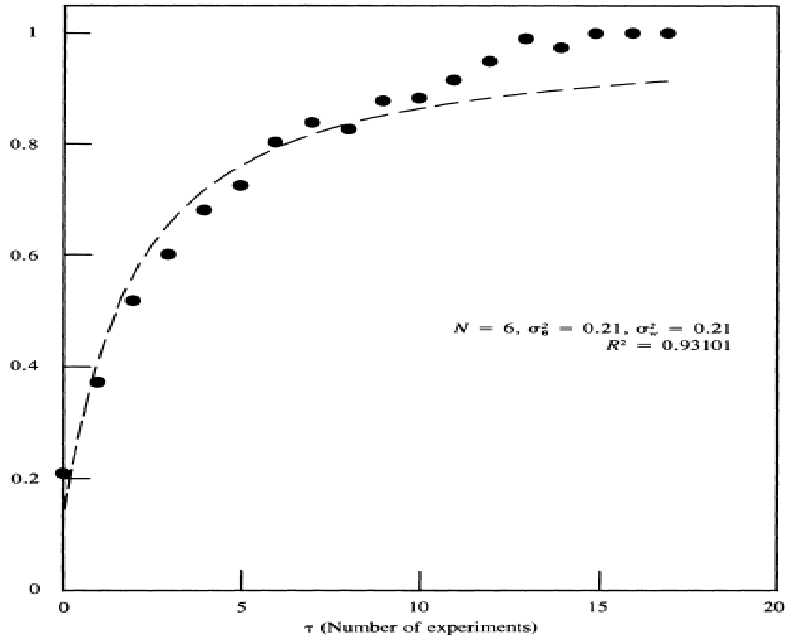

Figure 2: Bayesian processing of flight control empirical data
Source: Bayesian learning model (Javanovic & Nyarko, 1995)

The last two columns of the table show the upper and lower limits of success rates for the group, with the highlighted margins falling without actual success rate observed. Margin of error for the standard deviation of 3.21 was included to show that a higher standard deviation leads a higher error margin that includes more observations as highlighted. The Ito success rate coefficient of determination was better (R^2 = 0.96826) compared to the one obtained through the Bayesian method (R^2 = 0.93101).

Table 5: Actual and Ito processed success rates including lower and upper limits for 115 trainees

Number of Experiments	Actual success rate	Stochastic (Ito) success rate R^2 = 0.968	Margin of error (ME) for 40,000 runs (sd = 0.21)	ME for 115 runs (trainees) (sd = 0.21)	ME for 115 runs (trainees) (sd = 3.21)	Lower limit suc. rate for sd = 0.21	Upper limit suc. rate for sd = 0.21
1	0.2	0.2	0.000000	0.00000	0.00000	0.20000	0.20000
2	0.37	0.409011	0.000627	0.01649	0.47753	0.39252	0.42550
3	0.5	0.485779	0.000064	0.00030	0.10782	0.48548	0.48608
4	0.6	0.546118	0.000010	0.00281	0.04760	0.54331	0.54893
5	0.67	0.597335	0.000214	0.00585	0.14973	0.59149	0.60318
6	0.71	0.64251	0.000140	0.00085	0.09543	0.64166	0.64336
7	0.81	0.683303	0.000046	0.00345	0.06210	0.67985	0.68676
8	0.83	0.720727	0.000267	0.00443	0.04847	0.71629	0.72516
9	0.82	0.755459	0.000101	0.00245	0.02038	0.75301	0.75791
10	0.86	0.787976	0.000072	0.00292	0.05376	0.78506	0.79089
11	0.86	0.818631	0.000130	0.00186	0.03935	0.81677	0.82049

12	0.9	0.84769	0.000088	0.00363	0.05159	0.84406	0.85132
13	0.92	0.875365	0.000031	0.00321	0.00074	0.87215	0.87858
14	0.95	0.901822	0.000014	0.00447	0.02014	0.89736	0.90629
15	0.98	0.927199	0.000042	0.00118	0.05445	0.92602	0.92838
16	0.96	0.951609	0.000046	0.00140	0.00680	0.95021	0.95301
17	0.99	0.975147	0.000129	0.00128	0.00744	0.97387	0.97642
18	0.99	0.997894	0.000043	0.00052	0.04334	0.99738	0.99841

The operational standard deviation is the source of marginal error for the workers under examination. Consequently, the marginal error provides a basis for setting tolerance limits.

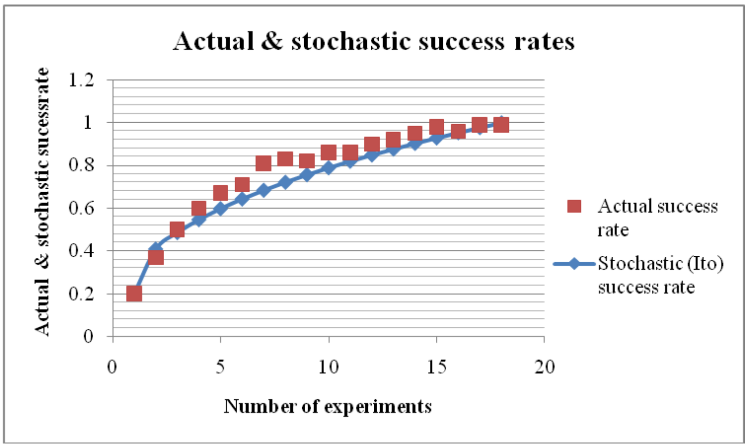

Figure 3: Visual fitting of actual and Ito processed success rates

Ito processing delivers a maximum of 0.984 as an approximated linear correlation. This translates to an R squared value of 0.96826 as shown in table 6.

Table 6: Actual & stochastic success rates correlates with increase in the number of experiments

Correlations: Actual vs. Stochastic success rate

		Actual success rate	Stochastic success rate
Actual success rate	Pearson Correlation	1	.984**
	Sig. (2-tailed)		.000
	N	18	18
Stochastic success rate	Pearson Correlation	.984**	1
	Sig. (2-tailed)	.000	
	N	18	18

**. Correlation is significant at the 0.01 level (2-tailed). $\mathbf{R^2 = 0.96826}$

Convexity, concavity and plateauing

Convexity refers to increasing marginal increase of the dependent variable usually plotted on the y axis, as the independent variable increases. Figure 1 depicts concavity, which is diminishing marginal increase of the dependent variable with increase in the independent variable. Exponential learning curves do exhibit these characteristics whose opposites can always be derived by plotting their reciprocal functions. These characteristics accord with the fact that additional learning of a specific task should go reducing with increased trials to a point where no more learning is expected. This is called plateauing; a situation where no more learning takes place such that if the curve manifested convexity, a vertical graph is traced for the rest of the range; otherwise, concavity traces a horizontal graph for the rest of the domain. Unfortunately, both modeling methods have not delivered plateaus by the 17th experiment. It is recommended that an additional function that delivers the plateau be formulated for appropriate portions of the domain so that the overall function becomes a two step function likely to be more comprehensive.

Conclusion

From deterministic exponential learning curves, it has been shown that stochastic equivalents can derive for both cumulative average time and marginal unit time versions. Not just to fit theoretical learning curves but also to fit empirical data. Moreover, stochastization by Ito processing provided a better fit for empirical data than the Bayesian approach. Both methods are Markovian, which is a basic characteristic ignored by deterministic learning curves. Further merits of Ito processing include provision of an interval estimate as opposed to a point estimate by utilizing the first two statistical moments – mean and standard deviation, besides availing productivity tolerance limits to the forecaster dependent on the groups' overall performance. This way, unique characteristics facing a particular cohort can be taken care of at the onset when the standard deviation of the group is determined after producing the first unit.

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R – Algorithm for the learning curve stochastization

```
Wt.1  <-  0
DeltaWt <- 0
DeltaWtlooper <- 0
timee <- 50
inters <- 20000
for(t in 1:timee)
{
Wt.1[1] = 1277
DeltaWtlooper <- 0
for( k in 1:inters)
{
DeltaWtlooper[k] <- -0.9409*0.2*Wt.1[t] + 0.97483*0.44721*Wt.1[t]* rnorm(1,0,1)
}
DeltaWt[t] <- mean(DeltaWtlooper)
Wt.1[t+1] <- Wt.1[t] + DeltaWt[t]
}
plot(1:(timee+1),(Wt.1), type="l", xlab="Time", ylab="Marginal Unit")
#Wealth at time t`
Wt.1[45]
```