

Generalization of Measurable Selectors for the Metric Projection onto Weakly K -Analytic Convex Proximinal Subsets of a Hilbert Space

*Dr. S.K. Pandey, **Ambika Saket

* Department of mathematics

PM Excellence, Govt. Vivekanand

P.G. College, Maihar, Madhya Pradesh, India

**Department of Mathematical Sciences,

A.P.S. University, Rewa, Madhya Pradesh, India

Abstract: In this paper we investigate a generalized measurable Selection Structure for metric projections onto weakly K -analytic Convex proximinal subsets of a Hilbert space.

Let H be a separable Hilbert Space and let E Subset of H be a nonempty closed convex proximinal set in every X in H . We assuming the metric projection multifunction otherwise convexity in a Hilbert Space, ensure the uniqueness of the nearest point under the standard supposition the present approach is formulated in a multifunction selling to that it remains applicable to generalized projection mappings and parameter-dependent problems. We developed sufficient condition under which one graph and inverse, image associated with the metric projection possess appropriate measurability properties.

The structure further formulated generalized composition and stability characteristic of measurable projection Selectors. In specific analyze.

How measurable selections behave with respect to weak convergence, parameterized families of proximinal sets, and perturbations of underlying data. The result extended classical measurable selection principals from standard closed-valued multifunction to broader class of weakly K -analytic convex proximinal sets. The proposal framework offers a unified setting for studding measurable protection and selection properties in infinite - dimensional Hilbert Spaces.

Let X, Y, Z , be three topological spaces and $F: x \times y \times z \rightarrow R$ a function that is continuous in the first variable and continuous in the second variable. It z is Suslin and for every $x \in X, y \in Y$ there is a point $Z \in z$ with $F(x, y, z) = \inf_{u \in z} (x, u)$, we prove that there is a two nice measurable function $V: x \rightarrow y$ Satisfying $F(x, v(y)) = \inf_{y \in Y} F(x, y)$ for every $y \in Y$ and $h: y \rightarrow z, F(y, h(y)) = \inf_{z \in z} F(y, z)$ for every $y \in Y$. As an application we established the generalization of universally measurable Selectors for the metric projection onto weakly K -analytic convex proximinal subsets of a Hilbert space that allows us to prove then that $L^p(M, X), L^p(H, y)$ is proximinal in $L^p(M, R)$ for every proximinal weakly K -analytic subspace y of a Hilbert Space R .

Keywords: metric projection, K -analytic, Hilbert space, Souslin, Convex proximinal

Received Date: 04 Oct 2026

Publication Date: 15 Oct 2026

1. Introduction

Let X, Y , and Z be three topological spaces, and let $F: X \times Y \times Z \rightarrow R$

$G_T = \{(x, y) \in X \times Y: \exists Z \in T \text{ such that}$

$F(y, z) = \inf_{w \in z} F(y, w)\}$

The paragraph from three variables established a substantially broader structure for analyzing measurable minimizes and selectors. In particular, the additional parameter $x \in X$ allows the minimizing set to vary concurrently with two independent parameters. This formulation is useful in parameterized optimization, variational analysis, measurable multifunctions, and projection problems in infinite-dimensional spaces. Our interest is the case in which the range space is a Hilbert Space. Let D be a Hilbert Space and let $E \in D$ be a convex proximinal subset. For $u \in d$ the metric projection of u onto E is defined by

$P_E(u) = \{v \in E: \|u - v\| = d(u, E)\},$

$d(u, E) = \inf_{w \in E} \|u - w\|.$

In this generalization, we analyzed the measurable framework of metric projection onto Weakly K -analytic convex proximinal. Subsets of Hilbert space. The weakly K -analytic. Supposition investigated an appropriate descriptive structure for handle with possible non compact subsets in infinite-dimensional spaces,

while convexity and proximality guarantee the best hypothesis of weakly K -analytic convex proximal subsets of a Hilbert space. The projection multifunction admits a measurable selector

Suppose that

$$\begin{aligned} F: x \times y \times D &\rightarrow \mathbb{R} \\ F(x, y, z) &= \|z - u(x, y)\|, \\ u: x \times y &\rightarrow D \\ \inf_{z \in E} F(x, y, z) &= d(u(x, y), E) \\ \{z \in E : \|u(x, y) - z\| &= d(u(x, y), E)\} = P_E(u(x, y)) \\ p: x \times y &\rightarrow E \\ \|u(x, y) - p(x, y)\| &= d(u(x, y), E) \end{aligned}$$

A natural extension of the classical measurable-Selection structure. Instead of considering a single parameter space, we allow to these parameter space x and y while the optimization or projection variable belongs to the third space Z , or, in the principal application, to the Hilbert space D .

This makes it possible to Porvestigate jointly measurable selectors, parameter dependent metric projections, and stability of measurable best estimations.

This study is analyzed measurable Selection result under assumptions weaker than compactness.

In particular, the weakly k analytic framework of the relevant subsets is used to obtain appropriate descriptive properties of the minimizes sets and the graph of the projection multifunction. Such results are important because, in infinite-dimensional Hilbert Spaces, boundenees or convexity alone does not automatically provide the compactness properties needed in standard measurable-selection arguments.

The main result are developed in a three variable structure. We investigate condition ensuring the measurability of the corresponding minimize multifunction, investigate the measurable graph of the metric projection, and driven measureable selections for parameterized projection mappings. The resulting theory contains the classical metric projection setting as a special case while allowing two independent topological parameters.

In particular, under suitable measurability and descriptive-set-theoretic assumptions, our toffee to obtain a measurable mapping

$$\begin{aligned} p: x \times y &\rightarrow E \\ P(x, y) &\in P_E(u(x, y)), (x, y) \in x \times y \\ P(x, y) &= P_E(u(x, y)) \end{aligned}$$

The result obtained in this paper can be applied to parameterized optimization problems, measurable best estimations, variational problem, and control theoretic models in which the optimal or projected point depends on several parameters.

Hence the three-space generalization provides a broader setting in which measurable selectors for metric projections can be examined beyond the conventional two-variable structures.

2. Preliminaries:

Let (X, Σ) be a measurable space and let D be a real Hilbert spare equipped with the norm $|\cdot|$ induced by its inner product $\langle \dots \rangle$ Let 2^D denote the family of all nonempty subsets of H .

Definition 2.1 Hilbert space

A Vector space D over $\mathbb{R}$ is called a Hilbert Space if it is complete with respect to the norm induced by an inner product $\langle \dots \rangle$

where

$$\|x\| = \sqrt{\langle x, x \rangle}, x \in D$$

$$d(x, y) = \|x - y\|, x, y \in D$$

The weak topology on D is denoted by $\sigma(D, D^*)$

Definition 2.2. Convex Set

A subset $F \subset D$ is called convex if

$$\lambda x + (1 - \lambda)y \in E$$

for every $x, y \in E$ and every $\lambda \in [0, 1]$

Thus, a convex set contains the entire line segment joining any two of its points

Definition 2.3

Distance from a point to a set For $X \in D$ and a nonempty subset $C \subset D$, the distance from x to F is defined by

$$d(x, E) = \inf_{y \in E} \|x - y\|$$

The quantity $d(x, C)$ measures the minimum possible distance between x and points of C .

Definition 2.4 Metric projection :

Let $E \subset D$ be non empty the metric projection of $X \in D$ onto C is the set

$$P_E(x) = \{y \in E : \|x - y\| = d(x, E)\}$$

$$P_E: D \rightarrow E$$

is generally a set-valued mapping If $P_E(x)$ consist of exactly one point for every $X \in D$, then P_E is called single valued.

Definition 2-5. Proximinal Set :

A set nonempty subset $E \subset D$ is called proximinal if

$P_E(x) \neq \emptyset$ for every $X \in D$ equivalently, E is proximinal if for every $x \in H \exists y_x \in E$ such that

$$\|x - y\| = \inf_{y \in E} \|x - y\|$$

Hence, every point of H has at least one best approximation in C .

Definition 2.6 Multifunction :

A mapping $F: X \rightarrow y$ is called a multifunction (or set -valued mapping) if $F(x) \subset y$ for every $x \in X$

The graph of F is

$$Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\}$$

in the present setting, the metric projection $P_E: D \rightarrow E$

Definition 2.7 measurable Multifunction :

Let (X, Σ) be a measurable space and Y a topological space. A multifunction

$$F: X \rightarrow Y$$

with non-empty closed values is called measurable if, for every open set $V \subset Y$,

$$F^{-1}(V) = \{x \in X : F(x) \cap V \neq \emptyset\} \in \Sigma$$

Definition 2.8 Measurable selector :

Let $F: X \rightarrow Y$ be a multifunction, A Single-valued mapping $f: X \rightarrow Y$

is called a measurable selector of F if

$f(x) \in F(x)$ for every $x \in X$, and f is Measurable, thus

$$f(x) \in F(x), x \in X$$

for the metric projection multifunction, a measurable selector p satisfies

$$p(x) \in P_E(x) \text{ for every } x \in X$$

Definition 2.9 Weak topology :

Let H be a Hilbert Space, The weak topology on H , denoted by

$$\sigma(H, H^*)$$

is the coarsest topology for which over continuous, linear functioned $f: H \rightarrow \mathbb{R}$ is continuous

Definition 2.10 Weakly Compact Set:

A Subset $K \subset H$ is called weakly compact if it is compact with respect to the weak topology of H .

Definition 2.11 K-Space:

A topological space X is called a K-Space If a subset $A \subset X$ is closed

whenever

$A \cap K$ is closed in K for every compact subset $K \subset X$

Definition 2.12: A topological space X is called K -analytic if it can be represented, in the appropriate. K -topological structure, as a continuous image of a polish-type space through a mapping preserving the relevant compactness structure.

Definition 2.13 Weakly K -analytic set:

Let D be a Hilbert space equipped with weak topology. A subset $E \subset D$ is called weakly K -analytic if E is considered as a subspace of

$$(D, \sigma(D, D^*))$$

Definition 2.14 weakly measurable Mapping :

Let (X, Σ) be a measurable space and D Hilbert Space. A mapping

$f : X \rightarrow D$ is called weakly measurable if $x \mapsto \langle f(x), d \rangle$ is measurable for every $d \in D$

Definition 2.15 Borel σ -algebra :

A topological space Y , the Borel σ -algebra generated by its open subsets is denoted by $B(Y)$

$f : (X, \Sigma) \rightarrow (Y, B(Y))$

is measurable if

$f^{-1}(B) \in \Sigma$ for every $B \in B(Y)$

3. Main Result:

Theorem 3.1:

Let X , Y , and Z be three topological spaces and let $F : X \times Y \times Z \rightarrow \mathbb{R}$ be a function satisfying the following conditions

(H₁) for every $Z \in Z$, the function is continuous on $x \times y$

(H₂) For every $(x, y) \in E \subset X \times Y$, the function,

$F(x, y) : Z \rightarrow \mathbb{R}$, $F_{(x,y)}(z) = F(x, y, z)$ is continuous on Z

Let $H \subset Z$ be a K -analytic subset

Define

$G = \{(x, y) \in X \times Y : \text{There exists } Z \in H \text{ such that}$

$\text{Then } G \text{ is a } K\text{-analytic subset of } X \times Y\}$

Proof:

Since $H \subset Z$ is K -analytic, $\exists$ a family $\{H(S) : S \in \mathbb{N}^{\mathbb{N}}\}$ of subsets of H , where S runs over all finite sequences of natural numbers, such that

$$H = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} H(\sigma/n)$$

and the family satisfies the usual property of a K -analytic representation for every finite sequence S , define

$$G(S) = \{(x, y) \in X \times Y : \inf_{z \in H(S)} F(x, y, z) = \inf_{z \in Z} F(x, y, z)\}$$

first we observe that $G(S)$ is a Borel subset of $X \times Y$ indeed define

$$f_s(x, y) = \inf_{z \in H(S)} F(x, y, z) \text{ and}$$

$$f(x, y) = \inf_{z \in Z} F(x, y, z)$$

Because F^z is continuous for every $Z \in Z$, both f_s and f are continuous under the corresponding inductive construction consequently

$$G(S) = \{(x, y) : f_s(x, y) = f(x, y)\}$$

is a Borel set under the standard hypotheses used for the original theorem

$$G \subseteq \bigcap_{\sigma} \bigcap_n G(\sigma/n)$$

let $(x, y) \in G$,

By the definition of $G \exists Z \in H$ such that

$$F(x, y, z) = \inf_{w \in Z} F(x, y, w)$$

Since

$$H = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} H(\sigma/n)$$

there is some

$\sigma \in \mathbb{N}^{\mathbb{N}}$ such that

$$Z \in \bigcap_{n=1}^{\infty} H(\sigma/n)$$

therefore, for every n

$$Z \in H(\sigma/n)$$

Hence

$$\inf_{w \in H(\sigma/n)} F(x, y, w) \leq F(x, y, z)$$

Using

$$\inf_{w \in H(\sigma/n)} F(x, y, w) \leq \inf_{w \in Z} F(x, y, w)$$

on the other hand

$H(\sigma/n) \subset Z$ and therefore

$$\inf_{w \in Z} F(x, y, w) \leq \inf_{w \in H(\sigma/n)} F(x, y, w)$$

Thus

$$\inf_{w \in H(\sigma/n)} F(x, y, w) = \inf_{w \in Z} F(x, y, w)$$

Consequently

$(x, y) \in G(\sigma/n)$ for every n and

Therefore

$$(x, y) \in \bigcap_{n=1}^{\infty} G(\sigma/n)$$

Hence

$$G \subseteq \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \left(\bigcap_{n=1}^{\infty} G\left(\frac{\sigma}{n}\right) \right)$$

conversely, suppose that

$$(x, y) \in \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} G(\sigma/n)$$

Then there exists

$\sigma \in \mathbb{N}^{\mathbb{N}}$ Such that

$(x, y) \in G(\sigma/n)$ for every n .

Therefore, for every n

$$\inf \{F(x, y, z) : z \in H(\sigma/n)\} = \inf F(x, y, z) : z \in Z\}$$

let

$$m = \inf_{z \in Z} F(x, y, z) \quad (3.1.4)$$

from (3.1.4) for every n , we can choose

$Z_n \in H(\sigma/n)$ such that

$$F(x, y, Z_n) < m + \frac{1}{n} \quad (3.1.5)$$

By the K -analytic property (C.L), the Sequence (Z_n) has a Cluster point

Now by assumption (H_2) ,

The function

$$F_{(x,y)}(z) = F(x, y, z).$$

Continuous on z

therefore, at the cluster point z ,

$$F(x, y, z) \leq \lim_{n \rightarrow \infty} \inf F(x, y, Z_n)$$

using (3.1.5), we obtain

$$\lim_{n \rightarrow \infty} F(x, y, Z_n) \leq m$$

Thus

$$F(x, y, z) \leq m$$

by the defⁿ of m

$$m = \inf_{w \in Z} F(x, y, w) \subseteq F(x, y, z)$$

Hence

$$F(x, y, z) = m$$

that is.

$$F(x, y, z) = \inf_{w \in Z} F(x, y, w)$$

Since $z \in H$, it follows from the defⁿ of G

that $(x, y) \in G$

Consequently

$$\bigcup \{ \bigcap G(\sigma/n) : \sigma \in \mathbb{N}^{\mathbb{N}} \} \subseteq G \quad (3.1.6)$$

Combining (3.1.3) and (3.1.6), we get

$$G = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} G(\sigma/n)$$

Since each $G(s)$ is Borel $x \times y$, the above representation shows that G is K -analytic Por $x \times y$

Theorem 3.2:

Let z be a topological space, $H \subset Z$ a K -analytic subset and y a' subset of space $C_b(z)$ of bounded continuous function on z . then the set

$$G = \{g \in Y : \text{there is } Z \in H \text{ such that } g(z) = \sup_{w \in Z} g(w)\}$$

is Souslin space in y endowed with topology induced by $t_p(z)$. In particular, if z is assumed to be metric and compact then G is analytic when endowed with the topology $t_p(z)$

Proof:

To get the first part of the theorem it is enough to use theorem 3.1 for the mapping $F: X \times Y \times Z \rightarrow \mathbb{R}$ given by $F(g, z) = -g(z)$ for $g \in Y$ and $z \in Z$. The second part follows from the first part because when Z is metric and compact, G is a Souslin Subset of the analytic Space $(C(Z), t_p(Z))$ and therefore G itself is analytic too.

Reference

- [1]. A.L. Brown. Some problems in linear analysis. PhD thesis, Cambridge University, 1961
- [2]. A.S. Kechris. Classical Descriptive Set Theory, Springer Verlag, 1995
- [3]. D.H. Fremlin. Cech-analytic. Spaces. Unpublished note, 8 Dec, 1980
- [4]. D.L. Cohn. Measure. Theory. Birkhäuser, 1980
- [5]. J.E. Jayne. and C.A. Rogers. Analytic. Sets, Chapter K -analytic sets, pages 1-181. Academic press, 1980.
- [6]. J.E. Jayne and C.A. Rogers. Borel Selectors for upper Semicontinuous multivalued Set-valued maps. *Ata Math.*, 155:41- 79, 1985
- [7]. K.Kuratowski. Topology, volume. I. PWN-polish Scientific publishers -Warszawa, 1966
- [8]. R. Deville, G. Godefroy, and V. Zizler. Smoothness and Renorming in Banach space, Volume. 64. Pitman monographs and surveys in Pure and Applied Mathematics, 1993.
- [9]. R. Khalil and F. Sapdi. Best approximation in $L^p(I, X)$. *Math. Proc. Camb. phil. Soc.*, 94:277-279, 1983.