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Measurable Selection theorems and Lifting Theorems for Multi-functions in Lusin and Metric Spaces

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Abstract: In this paper we have tried to study the theorems of measurable selector and measurable multifunction by using Suslin locally convex spaces, Hausdorff locally convex spaces, locally convex Metrizable separable vector spaces. The Kuratowski– Ryll- Nardzewski Measurable selector theorem is a classical theorem in measure theory that provides a sufficient condition for a multifunction to have a measurable selection. Let F: X→ P(X) be a mapping. Any function f: X → Y such that f(x)ɛ F(x) x ɛ X is called selector for F.
Keywords: Measurable Multifunctions, Seletor, Suslin type, Convex space, Metrizable separable.

In the present paper we have tried to investigate the existence of Caratheodory type selectors of multivalued mapping of two variables, measurable in one and continuous in other corresponding to multivalued mapping of two variables. 2000 Mathematics Subject Classification: Primary 20A20; Secondary 37A10.Key words: Souslin space, Caratheodory type selectors, (Iower semi continuous) mapping.

Introduction:

Since 1965 to 1980 the existence of single valued measurable selectors for multivalued mapping defined on single variable under the various possible condition measurability have been extensively studied by various authors : Aumann [1], Castaing [3], Debreu [5], Jacobs [10], Kuratowski and Nardzewski [11], Rockafellar [15], Van-Vleck [9],In addition to this, Michael [12], has tried to provide the condition for continuous selector for various miltivalued function in his three papers. Thereafter, these two properties. Measurability and continuity of multivalued mapping have been exploited in two different directions, one direction due to Himmelberg [8], where he had considered a single valued function of two variables that measurable in one and continuous in second and tried to show the existence. Of multivalued measurable mapping as well as its single selector for such measurable multivalued mapping in single variable. Where as in other direction, the existence of caratheodory type selectors began with the paper of Cellina [4] appeared in 1976. He had tried to show the existence of single valued measurable selectors of two variables, measurable in one and continuous in second corresponding to multivalued mapping defined for two variable satisfying the similar condition of measurability in one continuty in other. Thereafter many authors like Fryszowski [7] and Rybinski [14] have tried to extends this cellina [4] condition for some different topological space.

In this paper multivalued mapping of two variables has been defined on abstract measurable space along with souslin into a separable Banach space and Cratheodory selector has been derived of two variables measurable in one and l.s.c. in other. Notation and some supplementary Results. Through out this paper T will stand for an abstract measurable space with σ- algebra, A, X stands for Souslin space, Y is separable Banach space B (X) and B(Y) are Borel σ-field of subset of X and Y. Souslin space

2.1. A souslin space X is a Hausdorff topological space such that there exists a Polish space P and a continuous mafrom P on to X. Polish space

2.2. A topological space X is polish if it is separable and merizable by a complete metric.

Since Caratheodory type selectors are consequence of multivalued mapping of two variables such that it is measurable in one and l.s.c. in other, which is infact can be considered can be considered as the combination of following two theorems, where multivalued mapping have been defined for one variable having measurability and continuity condition separately.

Castanig theorem 2.3. if T is an abstract space, A is a σ field of subsets of T P maps T into closed subset of a polih space Y, then the measurability of P is equivalent to the existence of denumerable sequence {pn }of measurable selectors of P such that

P(t) = cl{pn (t)}n=1,2,….} for each tεT ,

Where cl srands for the closure. Michael Theorem 2.4. If X is locally compact metrizable and separable topological space, Y is a separable Banach space then P from X into closed and convex subsets of Y is lower semi continuous if there exists a sequence {pn } of continuous selectors of P such that P(t)=cl {pn (t) n=1,2,….} for each tɛT.

Now we consider following theorems proved by Himmelberg [8].

Theorem 2.5. Let T be complete, X souslin space, Y metric space,
f: T×X →Y measurable in t and continuous in x,and B a closed subset of Y. then t→F(t)={xɛX|f(t,x)εB} defined a measurable relation, in particular in f is real valued then t→{x|f(t,x) ≥ ƛ},t→{x/f(t,x)≤ƛ}, t→{x/f(t,x)=ƛ} are all measurable.

Theorem 2.6 L X be a separable metric space Y a metric space, f: T×X→Y a function measurable in t for each x and continuous in x for each t, F:T→X a measurable multifunction with complete values. Then the multi function G: T→Ydefined by G(t)=f(txF(t)) is weakly measurable.

SELECTION THEOREM MEASURABLE MULTIFUNCTIONS WITH VALUES IN COMPLATE SUBSETS OF A SEPARABLE METRIC SPACE

Definition: let $\Gamma: t\xi(z)$ a function $\sigma: t \longrightarrow x$ will be said to be a selection of Γ.
if $\sigma(t) \epsilon \tau(t)$ for every t.
An important problem is, when is non empty valued and has property of measurability, to prove existence of measurable selections.
The fundamental theorem is theoren 6, below, and its consequences, theorems 7 and 8.
But a very general and useful theorem is theoren 22.

Main result: 1

Theorem1 : −Let X be a separable metric srace (T, α) amamarable space, Γ a multifunction fron T to complete non empty subset a of R. If for each open set U in X.
$\Gamma^{-}(U)(= \{t|\Gamma(t) \cap U \neq \phi\})$ belongs to α . then Γ admits a measurable selection.

Proof: − Let $\{x_n\}$ be a countable dense set in X. We define a sequence of measurable functions assuming a countable number of values, (c_p) by recurrence, with the properties

$$d\left(\sigma_p(t)\Gamma(t)\right) < 2^{-p}\left(\sigma_{p+1}(t)\right), \sigma_p(t)) \leq 2^{-p+1}$$

First we put $\sigma_0(t) = x_n$ if n is the smallest integer such that
$\Gamma(t) \cap B\, n\, (x_n, 2^0) \neq \emptyset B(x_n, r)$ is the open ball of radius r and center x_n). Thus σ_0 is measurable:

$$\sigma_0^{-1}(x_n) = \Gamma\left(B(x_n, 2^0)\right) - \cup_{m<n}\Gamma^{-}(Bx_m, 2^0)).$$

Suppose now σ_p is chosen. Let $T_1 = \sigma_p^{-1}(x_1)$. Then if $t \epsilon (T_1)$.

$$\Gamma(t) \cap B(x_1, 2^{-p}) \neq \emptyset.$$

We put, on $T^1, \sigma_{p+1}(t) = x_n$
if a is the smallest integer such that $\Gamma(t) \cap B(x_1, 2^{-p}) \cap B\left(x_n, 2^{-(p+1)}\right) \neq \emptyset$.
Hence σ_{p+1} is mensurable, $d\left(\sigma_{p+1}(t), \Gamma(t)\right) < 2^{-(p+1)}$, and

$$d\left(\sigma_{p+1}(t), \sigma_p(t)\right) \leq 2^{-p} + 2^{-(p+1)} \leq 2^{-p+1},$$ *From the last inequality.*

follows that $\left(\sigma_p(t)\right)$ in a Cauchy equence. As $\Gamma(t)$ is complete e and $d\left(\sigma_p(t), \Gamma(t)\right) \to 0$, the limit of $\left(\sigma_p(t)\right)$ in the completion of X belongs to $\Gamma(t)$. This limit $\sigma(t)$ defines a measurable selection of Γ.

Main result: 2

Theorem2: − Under the name hypothesis as in theorem 6, there exista a sequence (σ_n) of measurable selections of Γ such that for every $t, \Gamma(t) = \{a_n(t)|n\, \epsilon\Gamma\}$.

Proof: − Let $[x_n]$ be a dense set in X. For $(n.1)\, \epsilon N^2$ set e

$$\Gamma_{ni}(t) = \{\Gamma^{(t)} \cap B(x_n, 2^{-i}) \text{ if } t \in \Gamma^{-}\left(B(x_n, 2^{-i})\right)\ \Gamma(t) \text{ therwise}.$$

The multifunction $t \to \Gamma_{ni}(t)$ has non empty complete values. For any open set U.

$$\{t|\Gamma_{ni}(t) \cap U \neq \emptyset\} = \{t\Gamma_{ni}(t) \cap \neq \emptyset\} = \Gamma^{-}(B(x_n, 2^{-i}) \cap U) \cup [\Gamma^{-}(B(x_n, 2^{-i})) \cap \Gamma^{-}(U)] \in \xi$$

Hence, by theorem 2. Γ_{ni} has a measurable selection σ_{ni}. Sow let us show that $\Gamma(t) = \{\sigma_{ni}(t)\}$. Let $x \in \Gamma(t)$ and $\varepsilon > 0$. Let i be such that $2^{-i} \leq \frac{\varepsilon}{2}$, and n such that $d(x_n, x) < 2^{-i}$.

Tt $\in \Gamma^{-}\left(B(x_n, 2^{-i})\right)$ *and* $\sigma_{ni}(t) \epsilon B(x_n, 2^{-i})$, hance, $d(\sigma_{ni}(t), x) \leq d(\sigma_{ni}(t), x_n) + d(x_n, x) \leq \varepsilon$.

Theorem : −2.1 let (T, ξ) a messurable space, X. a Polish apace, and Γ map T to non empty closed subsets of X. If for every open set U in X , $\Gamma^{-}(U) \epsilon\, \xi x$, then Γ Admite a sequence of measurable selections (σ_n) such that $\Gamma(t) = \{\sigma_n(t)\}$.

$$T\times\left[B\left(x_{n,}\frac{1}{p}\right)-\cup_{m<n}B(x_m,\frac{1}{p})\right],$$

Main Results: 3

MEASURABLE COMPACT CONVEX MULTIFUNCTIONS

Theorem 3:− *Let* (T,ξ) *be a measurable space,* E *a locally convex metrizable separable vector space, and* ε *map* T *to* $\wp_{ck}(E)$ *(the space of convex compact subsets of* X*). Then* Γ *is measurable if and only if the support functions* $\delta*(\Gamma(.))$ *are measurable.*

Proof. If Γ *is measurable,* $T_0=\{\Gamma(t)=\emptyset)\in\xi$. *If* $\delta*(\Gamma(.))$ *is measurable then*

$$T_0=\{t|\delta*(\Gamma(t))=-\infty\}\in\xi$$

Hence we may suppose $\Gamma(t)$ *non empty. If* Γ *is measurable, by theorem* 7 *there exists a sequence of mensurable selections* (σ_n),

such that $\Gamma(t)=[\sigma_n(t)]$ *so* $\delta*(\Gamma(.))=sup.<x',\sigma_n(t)>$

Is a measurable function of t.

Conversely suppose $\delta*(\Gamma(.))$ *measurable. There exists an increasing sequence of semi − norms* (p_n) *which define the topology of* E. *Let* h_n *be the Hausdorff semi distance associated with* p_n *By theorem* 11.12 *the distance on* $I_{ck}(E)$, $H(A,B)\Sigma 2^{-n}\ \frac{h_n(A.B)}{1+h_n(A,B)}$

defines the uniform structure of $I_{ck}(E)$. *We shall prove the following: for every* $A\in I_{ck}(E)$,

$t\to H(A,\Gamma(t))$ *is measurable.*

That, by lemma 16 *below (applied to* Γ *in place of* f *and* $I_{ck}(E)$ *in place of* Z *(see theorem II.*8*)), entails the theorem. It is sufficient to prove measurability of* $h_n(A,\Gamma(.))$. *But n by theorem II.*18,

$$h(A,B)=sup\{|\delta*(x'|A)-\delta*(x'|B)|\ x'\in v_n^0\}\ (with\ v_n=\{x|p_n(x)\le 1\}).$$

The set V_n^0 *is equicontinuous, hence compact for the topology of uniform convergence on compact subsets of* E. *Let* $\{x_n'\}$ *be a countable dense set in* V_n^0. *As* $\delta*(.|A)$ *and* $\delta*(.\Gamma(t))$ *are continuous for the above topology, we have*

$$h_n(A,\Gamma(t))=sup\ sup\ |\delta*(x_n'|A)-\delta*(x_n'|\Gamma(t))|.$$

Hence it is measurable in t.

Remark. When E *is a normed space, it suffices to prove that* $\forall x\in E$, $d(x,\Gamma(.))$ *is measurable.*

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