

Dynamics of Logistic Sequences: From Convergence to Chaos

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Abstract: The logistic difference equation is a fundamental model in population dynamics, capturing how a population evolves over discrete time under constraints such as limited resources. Despite its simple algebraic form, the logistic sequence exhibits a rich spectrum of behaviors ranging from stable equilibrium to periodic oscillations and chaotic dynamics. In this paper, we investigate the behavior of the logistic sequence defined by $p_{n+1} = kp_n(1 - p_n)$, where $0 < p_0 < 1$ and k is a positive parameter. Through numerical simulations and qualitative analysis, we explore how varying the parameter k influences long-term behavior. We demonstrate convergence to fixed points for small values of k , the emergence of periodic cycles for intermediate values, and the onset of chaos for larger values. Sensitivity to initial conditions is also examined, highlighting the unpredictability inherent in chaotic regimes. These findings illustrate how simple nonlinear systems can generate complex dynamics, with implications in ecology, mathematics, and dynamical systems theory.

Keywords: Logistic map, chaotic dynamics, period doubling, bifurcation diagram, nonlinear discrete systems

1. Introduction

Mathematical modeling plays a crucial role in understanding real-world phenomena, particularly in ecology where population dynamics are influenced by growth rates and environmental constraints. One of the most celebrated models in this domain is the logistic model, originally introduced to describe population growth under limited resources.

While continuous models such as differential equations are often used, discrete models provide a more realistic framework for populations that reproduce in distinct generations. The logistic difference equation is a classic example:

$$p_{n+1} = kp_n(1 - p_n)$$

Here, p_n represents the normalized population at generation n , scaled so that $0 \leq p_n \leq 1$, and k is a growth parameter.

What makes this equation particularly fascinating is that its behavior changes dramatically depending on the value of k . For certain values, the population stabilizes; for others, it oscillates; and in some cases, it behaves chaotically.

This paper aims to systematically investigate these behaviors through computational experiments and theoretical insights. The goals are:

- To determine conditions under which the sequence converges.
- To identify periodic behavior and bifurcations.
- To explore chaotic dynamics and sensitivity to initial conditions.

2. Mathematical Background

2.1 The Logistic Difference Equation

The logistic sequence is defined recursively:

$$p_{n+1} = kp_n(1 - p_n)$$

where:

- $p_n \in [0, 1]$
- $k \in (0, 4]$

This equation models reproduction (growth term kp_n and competition (limiting term $(1 - p_n)$)).

2.2 Fixed Points

A fixed point p^* satisfies:

$$p^* = kp(1 - p^*)$$

Solving:

$$p^* = (k(1 - p^*) - 1) = 0$$

Thus, fixed points are:

$$p^* = 0 \text{ and } p^* = 1 - \frac{1}{k}$$

2.3 Stability Analysis

The stability of a fixed point depends on the derivative:

$$f'(p) = k(1 - 2p)$$

- At $p^* = 0$: $f'(0) = k$
- At $p^* = 1 - \frac{1}{k}$: $f'(p^*) = 2 - k$

Stability condition: $|f'(p^*)| < 1$

This yields:

- $p^* = 0$ stable if $0 < k < 1$
- $p^* = 1 - \frac{1}{k}$ stable if $1 < k < 3$

3. Methodology

We conducted numerical simulations to compute sequences for various values of k and initial conditions p_0 . The procedure is as follows:

1. Choose a value of k
2. Select initial value $p_0 \in (0,1)$
3. Compute p_n iteratively
4. Plot and analyze behavior

For each scenario, we generated at least 100 terms to observe long-term dynamics.

4. Results and Analysis

4.1 Case 1: $1 < k < 3$ — Convergence to Equilibrium

For values such as $k = 2$, the sequence converges to a stable equilibrium:

$$p^* = 1 - \frac{1}{k}$$

For example:

- With $k = 2$, $p^* = 0,5$
Regardless of the initial value p_0 , the sequence approaches this limit.
Observations
- Convergence is smooth and monotonic (or slightly oscillatory)
- The limit is independent of p_0
- The system is predictable and stable

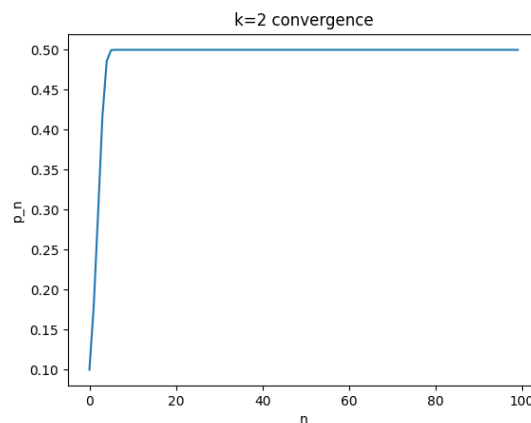


Fig 1: Convergence for $k=2$

4.2 Effect of Initial Conditions

We tested multiple initial values:

- $p_0 = 0.1$
- $p_0 = 0.5$
- $p_0 = 0.9$

All sequences converged to the same fixed point when $1 < k < 3$.

Conclusion

The long-term behavior is independent of initial conditions in this range.

4.3 Case 2: $3 < k < 3.4$ — Periodic Oscillations

When k exceeds 3, the fixed point becomes unstable.

Example: $k = 3.2$

The sequence oscillates between two values (period-2 cycle).

Observations

- Convergence to a repeating cycle
- No single limiting value
- Stable oscillation

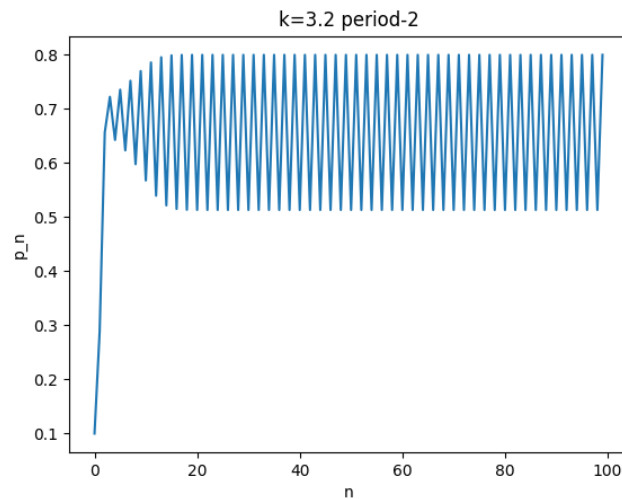


Fig 2: Period-2 oscillation for $k=3.2$

4.4 Case 3: $3.4 < k < 3.5$ — Period Doubling

As k increases further:

- Period-2 cycles become unstable
 - Period-4 cycles emerge
- This is known as period doubling bifurcation.

Observations

- Sequence cycles through 4 distinct values
- Increasing complexity
- Transition toward chaos

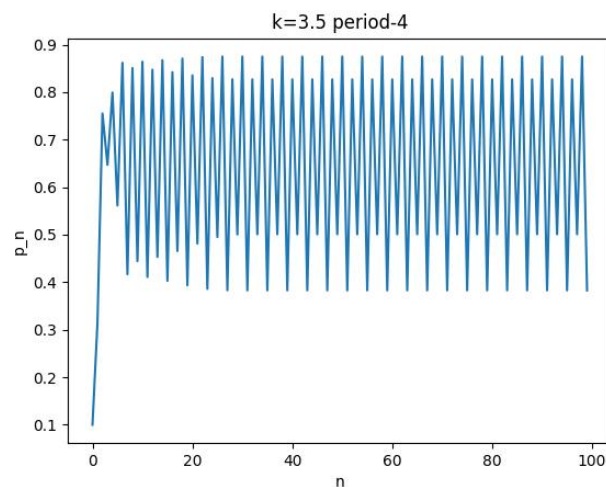


Fig 3: Period doubling for $k=3.5$

4.5 Case 4: $3.6 < k \leq 4$ — Chaos

For values such as $k = 3.8$ or $k = 4$, the behavior becomes chaotic.

Characteristics of Chaos

- No repeating pattern
- Extreme sensitivity to initial conditions
- Irregular fluctuations

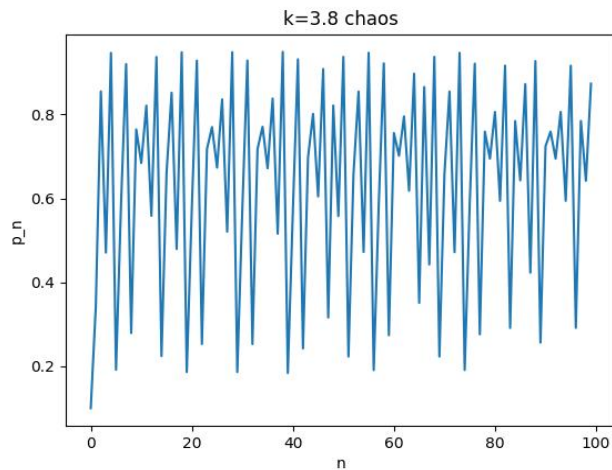


Fig 4: Chaos for $k=3.8$

4.6 Sensitivity to Initial Conditions

We tested:

- $p_0 = 0.500$
- $p_0 = 0.501$

For large k , the sequences quickly diverged.

Implication

Even tiny differences in initial conditions lead to vastly different outcomes.

This is a hallmark of chaotic systems.

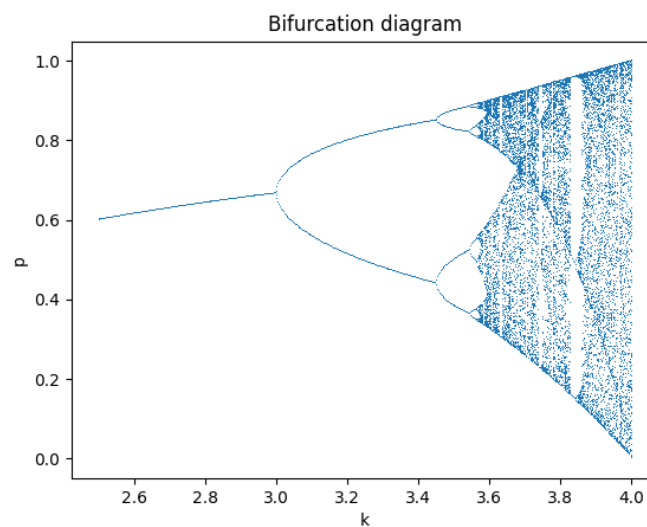


Fig 5: Bifurcation diagram

5. Discussion

5.1 Transition to Chaos

The logistic map demonstrates a gradual transition:

1. Stable equilibrium
2. Periodic oscillations
3. Period doubling
4. Chaos

This progression is a classic example of how deterministic systems can produce unpredictable behavior.

5.2 Ecological Interpretation

In ecological terms:

- Low k : stable population
- Moderate k : seasonal fluctuations
- High k : erratic population swings

This aligns with real-world observations in insect populations.

5.3 Mathematical Significance

The logistic map is a cornerstone of:

- Nonlinear dynamics
- Chaos theory
- Bifurcation analysis

It illustrates that complexity does not require complex equations.

5.4 Limitations

- The model assumes no external influences
- Real ecosystems are more complex
- Noise and environmental factors are ignored

6. Conclusion

The logistic difference equation provides a powerful framework for understanding complex dynamical behavior. Through numerical and theoretical analysis, we have shown that:

- For $1 < k < 3$, the system converges to a stable equilibrium.
- For $3 < k < 3.5$, periodic cycles emerge.
- For $k > 3.6$, the system becomes chaotic.
- Sensitivity to initial conditions is a defining feature of chaotic regimes.

These results highlight the richness of nonlinear systems and their relevance to real-world phenomena. The logistic map serves as a bridge between simple mathematical models and the complexity observed in nature.

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