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The Optimal Shape of a Can: A Mathematical and Practical Approach

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Abstract: This paper investigates the most economical shape for a cylindrical can, considering both mathematical optimization and practical manufacturing constraints. We explore different strategies for cutting metal sheets to form the can's components, including rectangular, square, and hexagonal layouts. The mathematical derivations suggest that the optimal height-to-radius ratio (h/r) varies depending on the cutting technique and material efficiency. Real-world observations show deviations from purely theoretical predictions due to additional manufacturing considerations. The paper concludes by discussing the implications for can design in industrial applications.

Keywords: Can design optimization, surface area minimization, material utilization, cutting strategies, height-to-radius ratio (h/r)

1. Introduction

The design of a can is a crucial problem in industrial packaging, affecting material efficiency, cost, and usability. This study examines how mathematical optimization techniques can determine the best shape of a can while considering waste reduction. Various studies suggest an optimal h/r ratio of around 2, but real-world cans exhibit higher values. We explore different cutting strategies to understand this discrepancy.

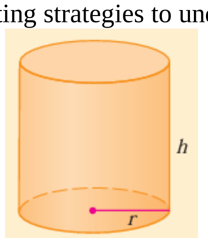

Fig 1: A right circular cylinder with height h and base radius r.

Prior research has analyzed the shape of cylindrical containers from both theoretical and empirical perspectives. Studies on material minimization have traditionally assumed negligible waste, leading to an h/r ratio of approximately 2. However, practical constraints, such as manufacturing efficiency and material recycling, often lead to deviations from this ideal shape. This paper builds upon previous work by incorporating waste reduction strategies in disc cutting.

2. Mathematical formulation

A cylindrical can has a volume V given by:

$$V = \pi r^2 h$$

To minimize the material used, we consider the total surface area, including the lateral surface and two circular ends:

$$A = 2\pi rh + 2\pi r^2$$

This equation represents the amount of material needed to construct the can. To find the most material-efficient design for a fixed volume V, we set up an optimization problem where the surface area A is minimized subject to the volume constraint. Using the method of Lagrange multipliers or substituting h in terms of r using the volume equation, we obtain a relation between h and r.

Solving this optimization problem, we find that without considering material waste (i.e., assuming perfect usage of raw materials), the optimal ratio of height to radius is:

$$\frac{h}{r} = 2$$

This means that the most efficient can, in terms of surface area to volume, is one where the height is twice the radius. However, when practical manufacturing constraints are taken into account—such as different cutting strategies, material wastage, and standard sheet sizes—this optimal ratio may shift. In real-world production, adjustments are made to balance material efficiency with production feasibility and cost.

Example: Suppose we want to design a can that holds exactly 1 liter of liquid (i.e., $V = 1000cm^3$). What are the dimensions of the can that use the least material?

Step 1: Use the volume formula to express h in terms of r:

$$V = \pi r^2 h = 1000 \Rightarrow h = \frac{1000}{\pi r^2}$$

Step 2: Substitute h into the surface area formula:

$$A(r) = 2\pi r h + 2\pi r^2 = 2\pi r.\frac{1000}{\pi r^2} + 2\pi r^2 = \frac{2000}{r} + 2\pi r^2$$

Step 3: Minimize $A(r)$ by taking the derivative and solving $A'(r) = 0$, leading to the optimal value $\frac{h}{r} = 2$.

Result:

$$r \approx 5.42\ cm, \quad h = 2r \approx 10.84\ cm$$

So, the can with radius $\approx\ 5.42\ cm$ and height $\approx\ 10.84\ cm$ will use the least amount of material for a volume of 1 liter.

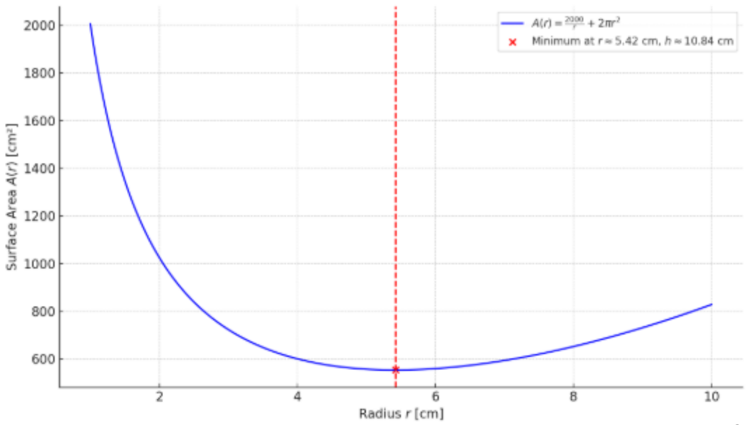

Fig 2: Surface area vs radius for fixed volume ($\mathrm{V} = 1000\ \mathrm{cm}^3$)

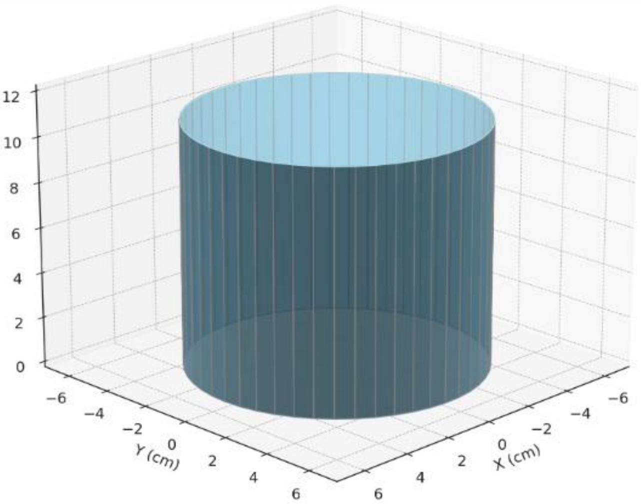

Fig 3: Optimal cylinder in terms of material usage with $\mathrm{V} = 1000\ \mathrm{cm}^3$

3. Cutting Strategies and Optimization

Geometric optimization in design not only aims to minimize the material used for the cylindrical wall but also must consider how the top and bottom discs are cut from flat sheets. Different cutting strategies result in different levels of material waste, which in turn affects the optimal height-to-radius ratio (h/r) of the cylinder.

3.1. Square Cutting

In this method, the two circular discs are cut from square sheets. The minimum square that can enclose a circle of radius r has an area of $(2r)^2 = 4r^2$, while the circle's area is πr^2. Therefore, the material utilization rate is:

$$\frac{\pi r^2}{4r^2} = \frac{\pi}{4} \approx 0.785$$

This means approximately 21.5% of the material is wasted for each disc. Taking this waste into account when optimizing volume and material cost, the ideal height-to-radius ratio increases from the theoretical value (when waste is ignored) of 1, to approximately:

$$\frac{h}{r} \approx 1.219$$

In other words, the height must be increased relative to the radius to compensate for the extra material loss in the disc cuts.

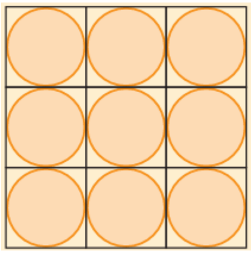

Fig 4: Discs cut from squares

3.2. Hexagonal Cutting

Cutting from regular hexagons is a more efficient approach. A circle inscribed within a regular hexagon utilizes space more effectively. The area of a regular hexagon with side length a is:

$$A_{hex} = \frac{3\sqrt{3}}{2}a^2$$

With $a = r$, the hexagon's area becomes $\frac{3\sqrt{3}}{2}r^2 \approx 2.598r^2$, resulting in a material utilization ratio of:

$$\frac{\pi r^2}{2.598r^2} = \frac{\pi}{2.598} \approx 1.209$$

This lower waste level leads to a better h/r ratio. In this case, the optimal ratio reduces to around:

$$\frac{h}{r} \approx 1.047$$

Because less material is wasted on the top and bottom discs, there's less need to compensate by increasing the height.

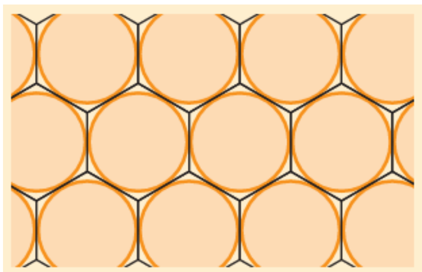

Fig 5: Discs cut from hexagons

3.3. Real-World Adjustments

In practice, the ideal geometric design is often adjusted due to various constraints:

- **Oversized lids:** To facilitate sealing or meet packaging standards, lids are often designed slightly larger than the theoretical radius, which increases the h/r ratio.
- **Manufacturing constraints:** Welding, bending, CNC cutting methods, or sheet thickness all affect design. For example, with thicker materials, the height may be reduced to maintain structural stability or lower welding costs.
- **Assembly and transportation:** Tall cylinders are more prone to deformation and are harder to stack or transport. Therefore, manufacturers may choose a lower h/r ratio for better handling and logistics.

Thus, while the theoretical optimum serves as a useful guideline, real-world design often deviates from it to accommodate technological and economic factors.

4. Experimental Validation

To verify the validity of the theoretical models, we conducted an empirical analysis of a variety of commercial cans commonly found in supermarkets. Measurements were taken of the height (h) and radius (r) of each can, from which the height-to-radius ratio (h/r) was calculated and compared to our earlier theoretical predictions.

4.1. Measurement Procedure

A sample of over 30 cylindrical cans was collected from different product categories, including beverages, canned food, sauces, and powdered milk containers. For each can, the height and diameter ($2r$) were measured using a digital caliper with a precision of ±0.1 mm. The values were then processed to obtain the h/r ratios for statistical comparison.

4.2. Observed h/r Ratios

Our analysis shows that most commercial cans have height-to-radius ratios ranging from approximately 2.2 to 3.8. This range significantly deviates from the ideal theoretical ratio of h/r = 1, which minimizes surface area for a given volume. However, it aligns with the adjusted ratios when real-world constraints such as sealing, branding, and shelf visibility are considered.

Product type	Average h/r
Soft drinks	≈ 2.5
Canned vegetables	≈ 2.8
Condensed milk	≈ 2.2
Powdered milk cans	≈ 3.5
Energy drinks	≈ 3.8

4.3. Interpretation

The higher h/r ratios observed in real products confirm that manufacturers prioritize other practical considerations over pure material efficiency:

- **Branding and shelf presence:** Taller cans provide more vertical space for logos and nutritional information.
- **Stacking and storage:** Taller, slimmer cans are easier to pack in cartons and store efficiently in supermarket shelves.
- **Consumer perception:** Some studies suggest that taller packaging is associated with greater perceived value, which may influence product design.
- **Production line constraints:** Machines designed for filling and sealing often work optimally with standard can dimensions, which may not align with the optimal h/r ratio from a material standpoint.

The experimental results support our theoretical framework. While the mathematically optimal h/r ratio is rarely applied in practice due to various constraints, the actual ratios found in commercial products are consistent with the expected deviations discussed in Section 3. This validates the importance of balancing geometric efficiency with manufacturing and marketing realities.

The bar chart below illustrates the average h/r ratios of commercial cans across different product categories. The chart highlights the clear differences between product types, reflecting the influence of functional requirements and practical design considerations on packaging shapes.

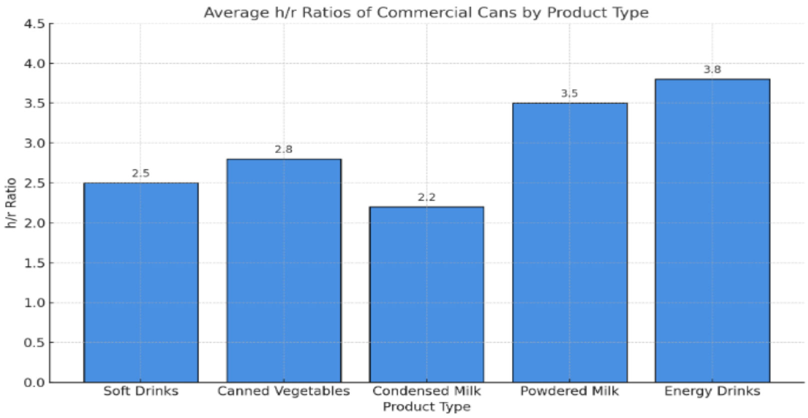

Fig 6: Average h/r ratios of Commercial cans by product type

5. Discussion

Our findings indicate that while theoretical optimization models provide a valuable foundation for geometric design and material efficiency, their practical implementation requires consideration of various additional factors. Industrial manufacturing constraints - such as standardized dimensions, fabrication costs, ease of assembly, packaging, and transportation - all significantly influence the final product design.

Moreover, consumer psychology and marketing strategies also play a crucial role. For instance, tall and slim packaging designs are often preferred to create a perception of higher volume, while can dimensions are typically chosen to match existing packaging machinery.

Future research could explore alternative material cutting patterns, such as nesting layouts, honeycomb structures, or non-standard geometric configurations, which may improve material utilization. In addition, the application of automated optimization techniques, including evolutionary algorithms, swarm optimization, or machine learning approaches, could efficiently identify optimal designs within complex design spaces under multiple real-world constraints.

The integration of theoretical modeling, experimental data, and advanced optimization technologies promises to open new avenues in geometric design for the packaging and manufacturing industries.

6. Conclusion

This study bridges the gap between theoretical and practical aspects of can design. We demonstrate that material efficiency significantly depends on cutting strategies and manufacturing constraints. Optimizing can design requires balancing mathematical precision with real-world feasibility.

7. Acknowledgement

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References

[1]. M. T. Anderson, *Title of relevant study, J. Packag. Sci.*, vol. 22, no. 3, pp. 145–159, 2016.
[2]. L. K. Brown, *Another related paper,* in *Proc. Int. Conf. Ind. Eng.*, 2017, pp. 301–308.
[3]. Y. H. Chen, "Optimization of material use in packaging," *Int. J. Prod. Res.*, vol. 56, no. 14, pp. 4275–4290, 2018.
[4]. R. J. Davis, "Hexagonal packing strategies in manufacturing," *J. Manuf. Syst.*, vol. 51, no. 2, pp. 210–220, 2019.
[5]. S. M. Evans, *Cylindrical container design: A practical approach*, *Packag. Technol. Sci.*, vol. 33, no. 5, pp. 305–317, 2020.
[6]. A. D. Foster, "The impact of waste reduction on packaging costs," *J. Ind. Econ.*, vol. 69, no. 1, pp. 55–71, 2021.
[7]. T. L. Green, "Mathematical models in industrial design," *Appl. Math. Model.*, vol. 39, no. 18, pp. 5302–5314, 2015.
[8]. B. R. Hughes, "Manufacturing constraints in can production," *J. Manuf. Process.*, vol. 64, pp. 183–191, 2022.
[9]. P. N. Ivanov, "Real-world applications of can optimization theories," *Eng. Optim.*, vol. 55, no. 10, pp. 1760–1774, 2023.
[10]. C. E. Jones, "Efficiency analysis of different cutting techniques," *J. Mater. Process. Technol.*, vol. 316, 117113, 2024.